Subject: Mechanics of Materials (MEC301)

Contents

Chapter-01 : FORCE

Defination of force:-

Force is defined as an external agency either push or pull which changes or tends to change the state of rest or uniform motion of a body upon which it act.

Unit of Force:-

→ S.I. unit of Force = Newton (N)

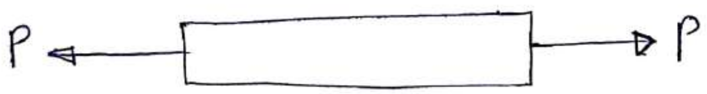
$$\boxed{1\,N = 1\,Kg\,m\,s^{-2}}$$

→ CGS unit of Force = Dyne

→ FPS unit of Force = poundal

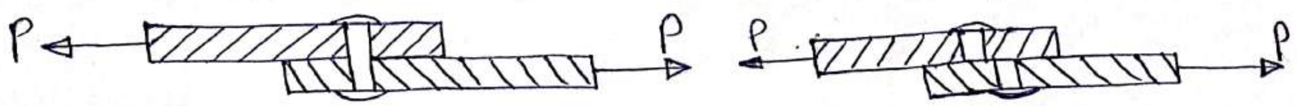
Types of forces

1. Tensile force:- The force which acts away from the point of application is called tensile force.



2. Compressive Force:- The force which acts towards the point of application is called compressive force.

3. Shear Force:- A force which acts parallel or tangential to the plane under consideration is called shear force.



Resolution of a Force:-

The way of representing a single force into number of forces without changing the effect of the force on the body is called as resolution of a force.

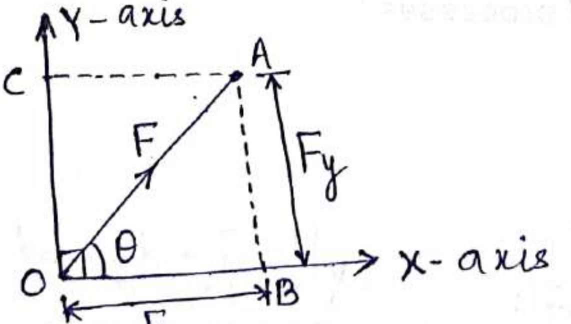
Methods of Resolution:-

There are two methods of resolution.

1. Resolution of a force into two mutually perpendicular components.
2. Resolution of a force into two non-perpendicular components.

1. Resolution of a force into two mutually perpendicular components

Let a force 'F' be inclined at an angle θ to x-axis as shown in figure. Length OA represent the magnitude of F. We have to resolve it into two components F_x along x-axis and F_y along y-axis



Draw perpendicular from A to point B on x-axis. Length OB represent the magnitude of x-component and AB represent the magnitude of Y- component.

In ΔOAB,

$$\cos\theta = \frac{OB}{OA}$$

$\therefore OB = OA\cos\theta$

But OA represents the force F in magnitude and direction.

$\therefore OB = F\cos\theta$

OB = x-component of force F

$\therefore$ $\boxed{F_x = \text{x-component of a force} = F\cdot\cos\theta}$

In ΔOAB,

$$\sin\theta = \frac{AB}{OA}$$

$\therefore AB = OA\sin\theta$

$\therefore AB = F\sin\theta$

AB = y-component of force F.

$\therefore AB = F\sin\theta$

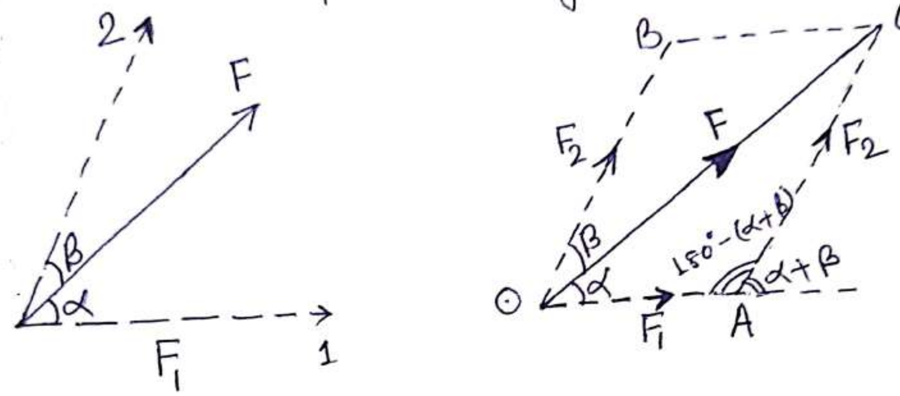
$\therefore AB = F_y = F\sin\theta$

$\boxed{F_y = \text{y-component of force } F = F\cdot\sin\theta}$

*2. Resolution of a force into two non-perpendicular components:-

Let F_1 and F_2 be the components of F along axes at angles α and β with F.

Complete the parallelogram OACB.



In $\triangle$ OAC, applying sine rule,

$$\frac{F_1}{\sin\beta} = \frac{F_2}{\sin\alpha} = \frac{F}{\sin[180° - (\alpha+\beta)]}$$

$$\therefore \quad \frac{F_1}{\sin\beta} = \frac{F_2}{\sin\alpha} = \frac{F}{\sin(\alpha+\beta)}$$

$$\therefore \quad \boxed{F_1 = \frac{F\sin\beta}{\sin(\alpha+\beta)}}$$

$$\therefore \quad \boxed{F_2 = \frac{F\sin\alpha}{\sin(\alpha+\beta)}}$$

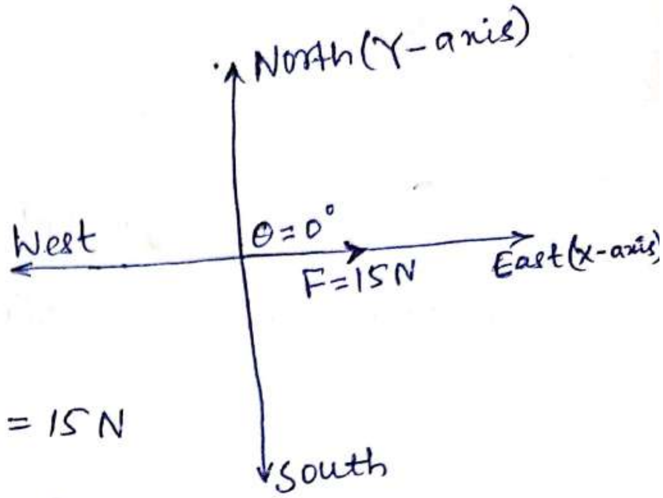
Q1 Resolve a force of 15 N acting due East or horizontally towards right or along positive x-axis.

Solution:-

Given:- $F = 15N$

$\theta = 0^\circ$

Find:- F_x and F_y.



We know that

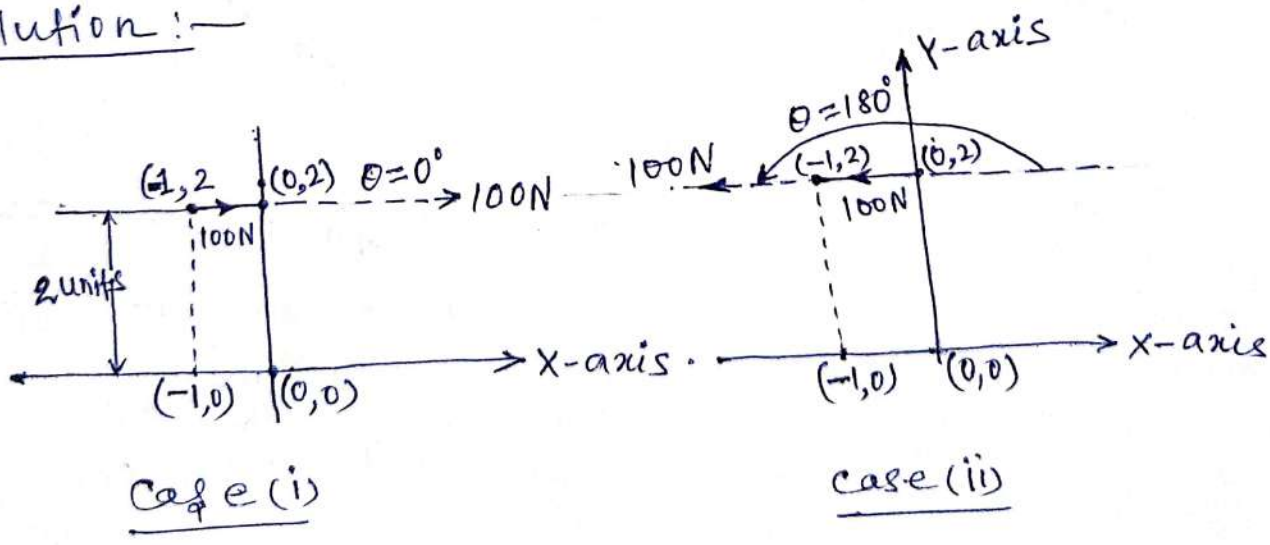
$F_x = F\cos\theta = 15\cos 0^\circ = 15 \times 1 = 15\,N$

$F_y = F\sin\theta = 15\sin 0^\circ = 15 \times 0 = 0$

$\boxed{F_x = 15N,\ F_y = 0}$ Ans

Q2 Find the components of force 100N passing through the points (0,2) and (−1,2).

Solution:-



Case (i):- If the force acts from (−1,2) towards (0,2) i.e. horizontally towards right, $\theta = 0^\circ$ from +ve x-axis.

$\therefore F_x = F\cos\theta = 100\cos 0^\circ = 100 \times 1 = 100\,N$

$\therefore F_y = F\sin\theta = 100\sin 0^\circ = 100 \times 0 = 0$

Case (ii): If the force acts from (0, 2) towards (−1, 2) i.e. horizontally towards left, $\theta = 180°$ from +ve X-axis.

$\therefore F_x = F\cos\theta = 100\cos 180° = 100 \times (-1) = -100\,N$

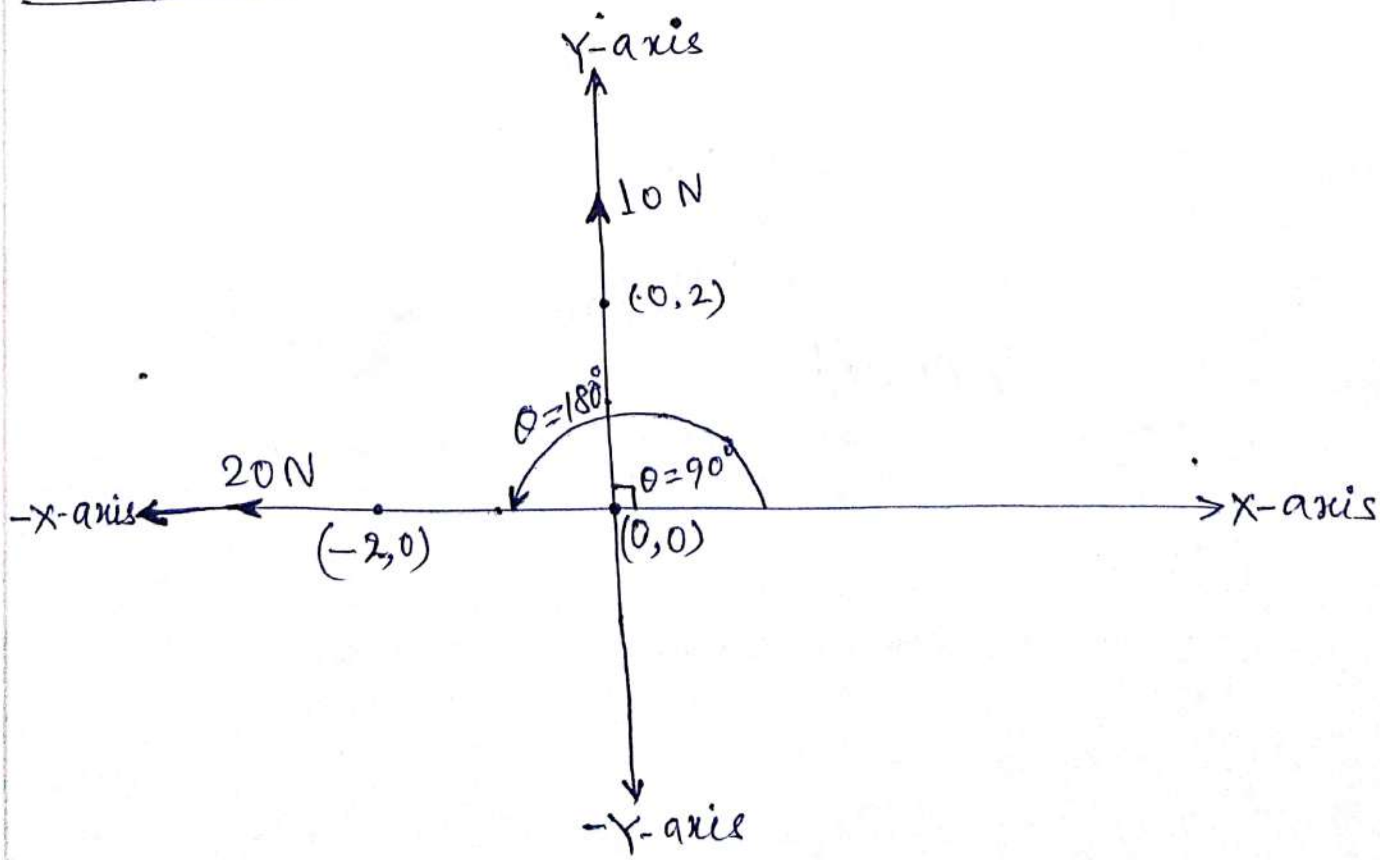
$\therefore F_y = F\sin\theta = 100\sin 180° = 100 \times 0 = 0$

case (i)	$F_x = 100\,N$, $F_y = 0$
case (ii)	$F_x = -100\,N$, $F_y = 0$

Ans

Q3. If two forces 10 N and 20 N are passing through origin and other points (0, 2) and (−2, 0) respectively, what are the rectangular components of these two forces?

Solution:—



Find:- Retangular components F_x and F_y

Let us assume that both the forces (10 N and 20 N) are pull. i.e. They act away from the origin.

(i) Components of 10 N passing through (0,0) and (0,2)

Here, $F = 10\ N$, $\theta = 90^\circ$ from +ve x-axis

We know that,

$$F_x = F\cos\theta = 10\cos 90^\circ = 10 \times 0 = 0$$

$$F_y = F\sin\theta = 10\sin 90^\circ = 10 \times 1 = 10\ N$$

(ii) Components of 20 N passing through (0,0) & (-2,0)

Here, $F = 20N$, $\theta = 180^\circ$ from +ve x-axis

we know that,

$$F_x = F\cos\theta = 20\cos 180^\circ = 20 \times (-1) = -20\ N$$

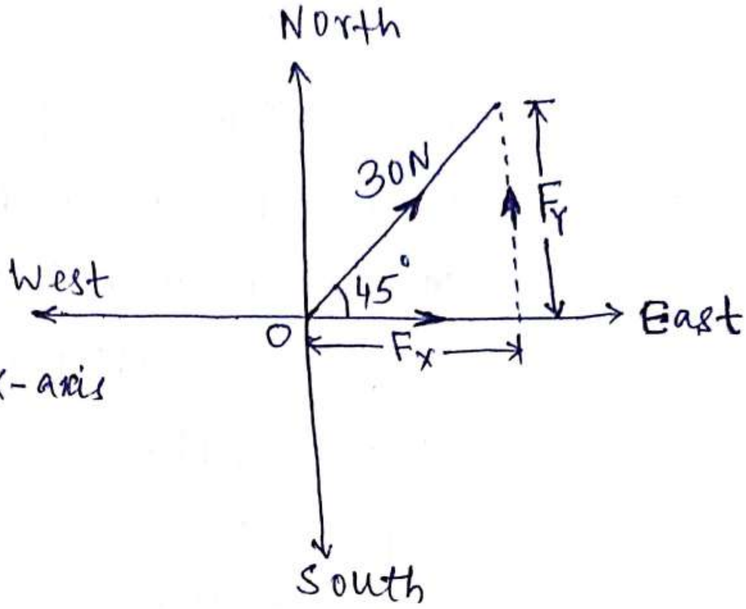
$$F_y = F\sin\theta = 20\sin 180^\circ = 20 \times 0 = 0$$

10 N force :	$F_x = 0$, $F_y = 10\ N$
20 N Force :	$F_x = -20N$, $F_y = 0$

Ans

(04). Resolve a force of 30 N acting North-East away from the point.

Solution:-



~~Soln~~:

Given:- $F = 30\,N$

θ = Angle made by force with +ve x-axis
$= 45^\circ$

We know that,

$F_x = F\cos\theta$

$= 30\cos 45^\circ$

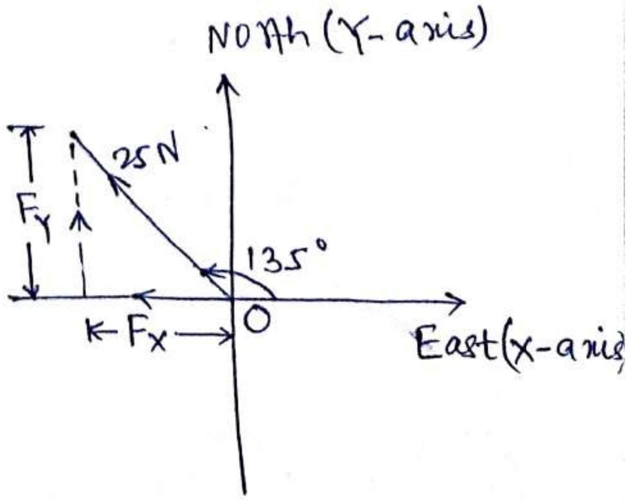
$F_x = 21.21\,N$

$\therefore F_y = F\sin\theta = 30\sin 45^\circ = 21.21\,N$

$$\boxed{F_x = 21.21\,N \quad \text{and } F_y = 21.21\,N}$$ Ans

(05) Resolve a force of 25 N acting North-West away from the point.

Solution:-



Given:- $F = 25\,N$

θ = Angle made by force with +ve X-axis
$= 135^\circ$

Find:- F_x and F_y.

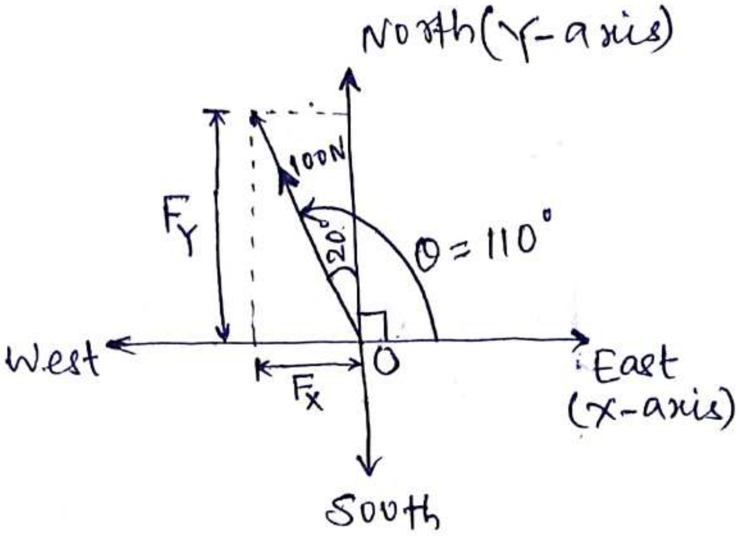
We know that,

$F_x = F\cos\theta = 25\cos 135^\circ = -17.68\,N$

$F_Y = F\sin\theta = 25\sin 135^\circ = 17.68\,N$

$$\boxed{\begin{aligned} F_x &= -17.68\,N \\ F_y &= 17.68\,N \end{aligned}}$$ Ans

(06) Resolve a force of 100N acting at 20° West-North (or 20° West of North) away from the point.



Solution:-

Given:- $F = 100\,N$

θ = Angle made by force with x-axis

$= 90^\circ + 20$

$= 110^\circ$

Find:- F_x and F_y.

we know that,

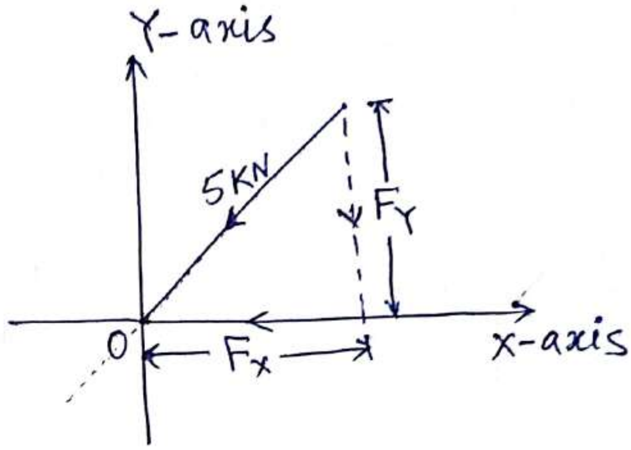
$F_x = F\cos\theta = 100\cos 110^\circ = -34.2\,N$

$F_Y = F\sin\theta = 100\sin 110^\circ = 93.97\,N$

$$\boxed{\begin{aligned} F_x &= -34.2\,N \\ F_Y &= 93.97\,N \end{aligned}}$$ Ans

07 Resolve a force of 5KN inclined at 30° with x- axis acting ~~towards~~ the point.

Soln:-



Given:- $F = 5 KN$

θ = acute angle made by force with +ve x-axis = 30°

Find:- F_x and F_y.

We know that

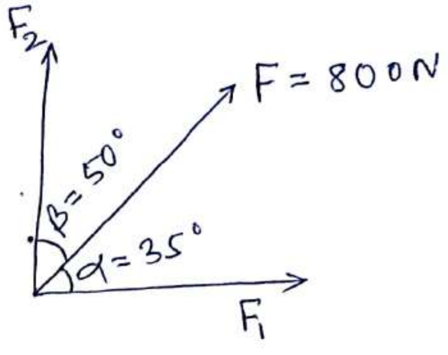
$$F_x = -F\cos\theta = -5\cos 30^\circ = -4.33 kN$$

$$F_y = -F\sin\theta = -5\sin 30^\circ = -2.5 KN$$

$$\therefore \boxed{\begin{aligned} F_x &= -4.33 KN \\ F_y &= -2.5 KN \end{aligned}}$$

Q8: A force of 800 N is acting at a point. Resolve this force along 35° on its one side and 50° on its another side.

Solution :-



Given; $F = 800\ N$
$\alpha = 35°$
$\beta = 50°$

Find: F_1 and F_2.

Let F_1 = Component of F along direction 1

$$\therefore F_1 = \frac{F \sin\beta}{\sin(\alpha+\beta)}$$

$$\therefore F_1 = \frac{800 \sin 50°}{\sin(35°+50°)}$$

$$\therefore F_1 = 615.18\ N$$

Let, F_2 = Component of F along direction 2

$$\therefore F_2 = \frac{F \sin\alpha}{\sin(\alpha+\beta)}$$

$$\therefore F_2 = \frac{800 \sin 35°}{\sin(35°+50°)}$$

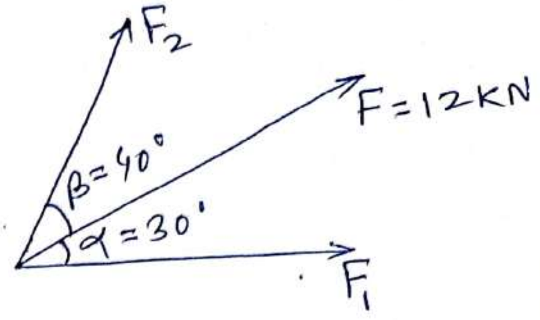
$$\therefore F_2 = 460.61\ N$$

F_1 along $35° = 615.18\ N$
F_2 along $50° = 460.61\ N$

Ans

Q9. Resolve the force of 12 KN in two directions at 30° and 40° on either side of it.

Solution:-



Given: $F = 12\,KN$
$\alpha = 30°$
$\beta = 40°$

Find: F_1 and F_2.

We know that

$$F_1 = \frac{F \sin\beta}{\sin(\alpha+\beta)}$$

$$\therefore\ F_1 = \frac{12 \sin 40°}{\sin(30° + 40°)}$$

$$\therefore\ F_1 = 8.21\,KN$$

and

$$F_2 = \frac{F \sin\alpha}{\sin(\alpha+\beta)}$$

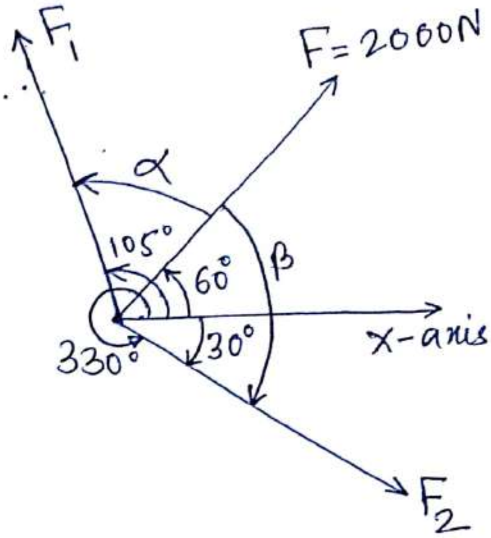
$$\therefore\ F_2 = \frac{12 \sin 30°}{\sin(30° + 40°)}$$

$$\therefore\ F_2 = 6.38\,KN$$

F_1 along $30° = 8.21\,KN$
F_2 along $40° = 6.38\,KN$

Ans

(10). A force of 2000 N acts at an angle of 60° with x-axis. Find its components along 105° and 330° with x-axis.



Solution: —

Given: F = 2000 N

Find: - F_1 & F_2.

α = Angle between F and F_1,

$\alpha = 105° - 60° = 45°$

$\boxed{\alpha = 45°}$

Angle between x-axis and $F_2 = 360° - 330° = 30°$

∴ β = Angle between F and F_2

∴ $\beta = 60° + 30° = 90°$

$\boxed{\beta = 90°}$

We know that,

$F_1 = \dfrac{F\sin\beta}{\sin(\alpha+\beta)}$

$F_1 = \dfrac{2000\sin 90°}{\sin(45°+90°)}$

$F_1 = \dfrac{2000 \times 1}{\sin 135°}$

$F_1 = 2828.43\text{ N}$

$F_2 = \dfrac{F\sin\alpha}{\sin(\alpha+\beta)}$

$F_2 = \dfrac{2000\sin 45°}{\sin(45°+90°)}$

$F_2 = \dfrac{2000\sin 45°}{\sin 135°}$

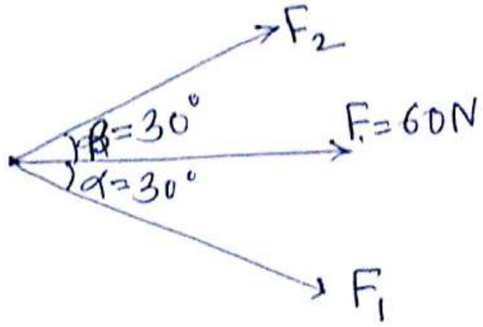
$F_2 = 2000\text{ N}$

F_1 along 105° = 2828.43 N
F_2 along 330° = 2000 N

Ans

(11). What are the components of 60 N force acting horizontal, in two directions on either side, at an angle 30° each ?

Solution:-



Given: $F = 60\ N$.
$\alpha = 30^\circ$
$\beta = 30^\circ$

Find: F_1 and F_2

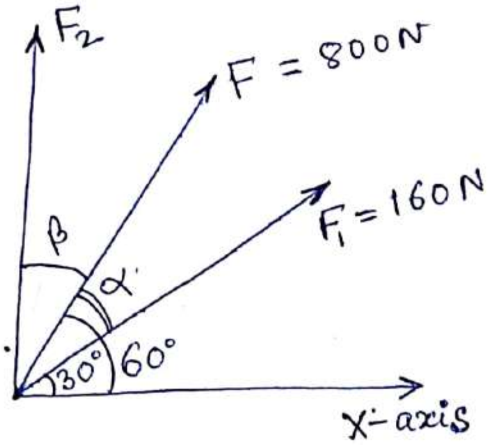
We know ~~that~~

$F_1 = \dfrac{F \sin\beta}{\sin(\alpha+\beta)}$	$F_2 = \dfrac{F \sin\alpha}{\sin(\alpha+\beta)}$
$\therefore F_2 = \dfrac{60 \sin 30^\circ}{\sin(30^\circ + 30^\circ)}$	$\therefore F_2 = \dfrac{60 \sin 30^\circ}{\sin(30^\circ + 30^\circ)}$
$\therefore F_2 = \dfrac{60 \sin 30^\circ}{\sin 60^\circ}$	$\therefore F_2 = \dfrac{60 \sin 30^\circ}{\sin 60^\circ}$
$\therefore F_2 = 34.64\ N$	$\therefore F_2 = 34.64\ N$

$$\boxed{\begin{array}{l} F_1 \text{ along } 30^\circ = 34.64\ N \\ F_2 \text{ along } 30^\circ = 34.64\ N \end{array}}$$ Ans

(12) The resultant of two forces in a plane is 800 N at 60° with X-axis. One force is 160 N at 30° with X-axis. Determine the missing force and its inclination.

Solution :-



Given:- $F = R = 800\,N$

$F_1 = 160\,N$

Angle between F and X-axis = 60°

Angle between F_1 and X-axis = 30°

Angle between F and F_1 = 60° − 30°

$$\boxed{\alpha = 30^\circ}$$

Find:- Angle between F and $F_2 = \beta$.

and value of F_2.

We know that,

$$F_1 = \frac{F \sin\beta}{\sin(\alpha+\beta)}$$

$$\therefore \quad 160 = \frac{800 \sin\beta}{\sin(30^\circ+\beta)}$$

$$\therefore \quad 160 \sin(30^\circ+\beta) = 800 \sin\beta$$

$$\therefore \quad 160\,[\sin 30^\circ \cos\beta + \cos 30^\circ \sin\beta] = 800 \sin\beta$$

$$\therefore \quad 160\,(0.5 \cos\beta + 0.866 \sin\beta) = 800 \sin\beta$$

$$\therefore \quad 80 \cos\beta + 138.56 \sin\beta = 800 \sin\beta$$

$$\therefore \quad 80 \cos\beta = (800 - 138.56) \sin\beta$$

$$\therefore \quad 80 \cos\beta = 661.44 \sin\beta$$

$$\therefore \quad \frac{\sin\beta}{\cos\beta} = \frac{80}{661.44} = 0.121$$

$\therefore \tan\beta = 0.121$

$\therefore \beta = \tan^{-1}(0.121)$

$\therefore \boxed{\beta = 6.89^\circ}$

We know that,

$$F_2 = \frac{F \sin\alpha}{\sin(\alpha+\beta)}$$

$$\therefore F_2 = \frac{800 \sin 30^\circ}{\sin(30^\circ + 6.89^\circ)}$$

$$\therefore F_2 = \frac{800 \times 0.5}{\sin 36.89^\circ}$$

$$\therefore F_2 = \frac{400}{0.6} = 666.67\ N$$

$\boxed{F_2 = 666.67\ N \text{ and } \beta = 6.89^\circ}$ Ans

Chapter – 02
Load and Stress-Strain

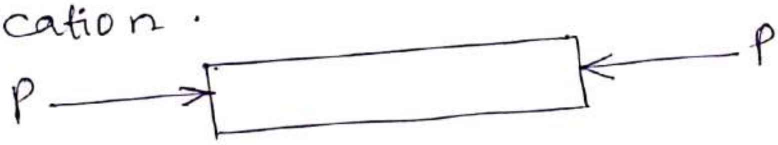
Load:- External forces and moments (Couples) acting on a body are called load.

Types of loads:-

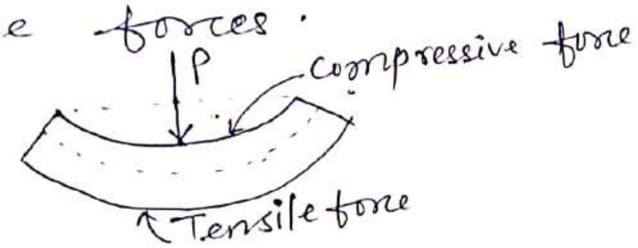
There are basically four types of loads.

① **Tensile loads**:- Tensile loads are the forces which acts away from the point of application.

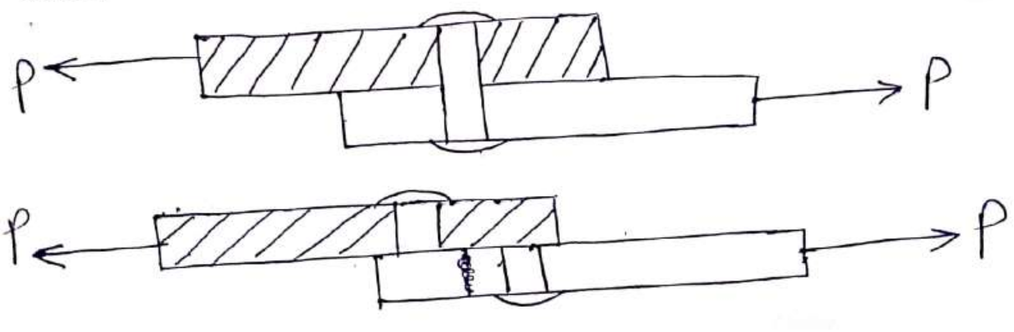
② **Compressive loads**:- Compressive loads are the forces which acts towards the point of application.



③ **Bending loads**:- Bending loads are the combination of tensile forces and compressive forces.



④ **Shear loads**:- Shear loads are the forces which acts parallel or tangential to the plane under consideration.



- Unit of stress is also denoted by Kilopascal (KPa), Megapascal (MPa) and Gegapascal (GPa).

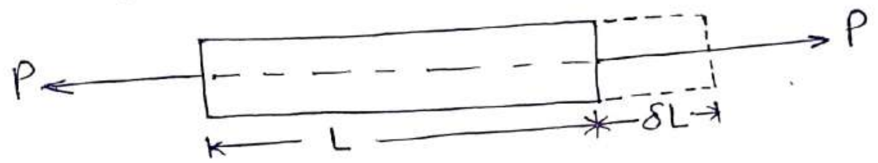
$1\ KPa = 10^3\ Pa = 10^3\ N/m^2$

$1\ MPa = 10^6\ Pa = 10^6\ \frac{N}{m^2} = \frac{10^6 N}{10^6 mm^2} = 1\ N/mm^2$

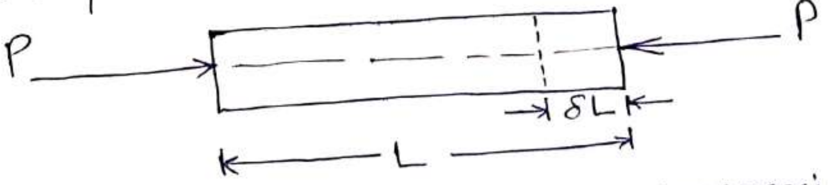
$1\ GPa = 10^9\ Pa = 10^9\ \frac{N}{m^2} = \frac{10^9 N}{10^6 mm^2} = 10^3\ N/mm^2 = 1\ KN/mm^2$

Types of stress :—

① Tensile stress :- When two equal and opposite tensile force applied to a body tend to elongate it, the body is said to be under tension and the stress induced in it is called as tensile stress.



② Compressive stress :- When two equal and opposite compressive forces applied to a body tend to shorten it, the body is said to be under compression and the stress induced in it is called compressive stress.



Here, compressive stress $= \frac{\text{Compressive force}}{\text{cross-sectional Area}}$

$\sigma_c = P/A$

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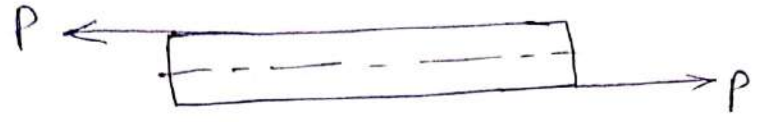
③ Shear stress:- When two equal and opposite forces acting tangentially to the section of a body tends to slide its one part over the other part, the body is said to be a state of shear & the stress induced in it is called shear stress.

→ Shear stress is denoted by 'τ'.

→ Shear stress = $\frac{\text{Shear force}}{\text{cross-sectional Area}}$

$$\tau = P/A$$



* Strain:- The ratio of change in length of a member to the original length is called as strain.

→ It is denoted by 'e'.

$$\text{strain} = \frac{\text{change in length}}{\text{original length}}$$

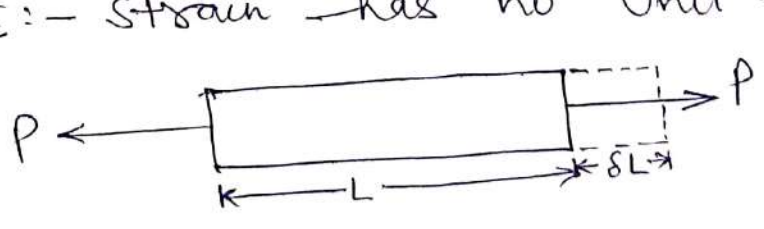
$$\boxed{e = \frac{\delta L}{L}}$$

Where, e = strain

δL = change in length

L = original length

Unit:- Strain has no unit.



* Types of Strain :-

There are basically three types of strain.

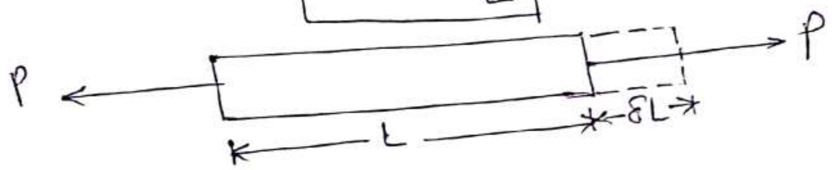
1. Tensile strain
2. Compressive strain
3. Shear strain

(1) **Tensile strain:**– The ratio of increase in length to the original length of a body due to tensile force is called as tensile strain.

→ It is denoted by e_t.

$$\text{Tensile strain} = \frac{\text{increase in length}}{\text{original length}}$$

$$\boxed{e_t = \frac{\delta L}{L}}$$

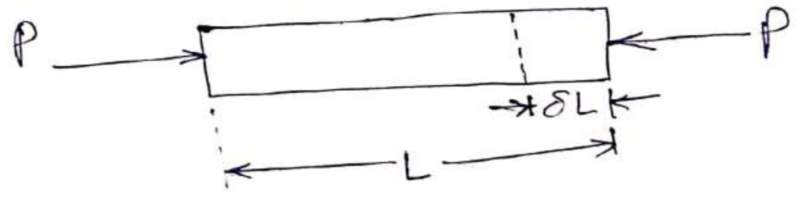


(2) **Compressive strain:** – The ratio of decrease in length to the original length of a body due to compressive force is called as compressive strain.

→ It is denoted by e_c.

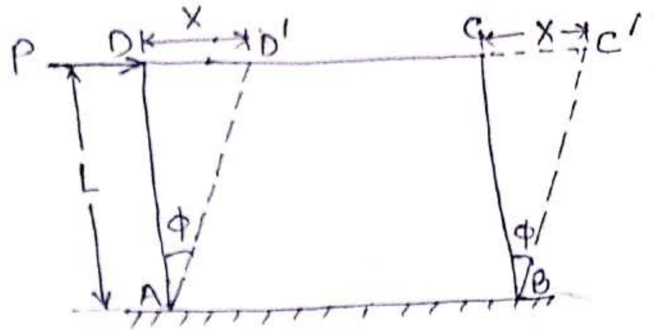
$$\therefore \text{Compressive strain} = \frac{\text{Decrease in Length}}{\text{Original Length}}$$

$$\boxed{e_c = \frac{\delta L}{L}}$$



(3) **Shear strain:**– The angular deformation produced in member due to shear force is called as shear strain.

Here ABCD is a cube.
Side of cube = L
AB Face is fixed.



NOW, $\tan\phi = \frac{DD'}{AD} = \frac{X}{L}$

$$\boxed{\phi = \frac{X}{L}}$$

$\therefore$ ϕ is very small angle. [$\tan\phi \approx \phi$]

Here, ϕ = Shear strain
X = shear deformation.
L = cube side length.

* HOOK'S LAW

It states "when a material is loaded within its elastic limit, the stress produced is directly proportional to the strain".

Mathematically,
stress $\propto$ strain

$\therefore$ stress = constant × strain

$$\therefore \boxed{\frac{\text{stress}}{\text{strain}} = \text{constant}} \quad \text{—— (i)}$$

The ratio of stress and strain which is a constant within the elastic limit is called modulus of elasticity.

→ Modulus of elasticity is denoted by 'E'.

$$\boxed{\frac{\sigma}{e} = E} \quad \text{(From equation (i))}$$

where, σ = stress
e = strain
E = Modulus of elasticity.

<u>Young's Modulus</u> :- The ratio of stress and Strain within the elastic limit is called as Young's Modulus.

→ Young's Modulus is also called as Modulus of Elasticity.

→ Young's Modulus is denoted by 'E'.

→ S.I. unit of Young's Modulus is N/m^2.

$$\cancel{\frac{Stress}{Strain}}$$

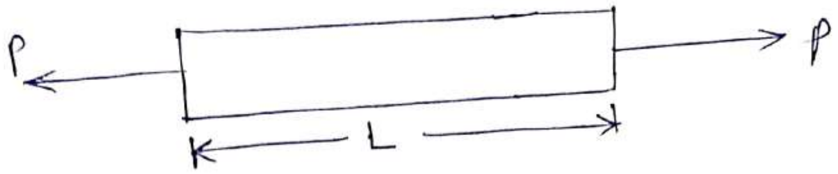
$$\text{Young's Modulus} = \frac{\text{Stress}}{\text{Strain}}$$

$$\boxed{E = \frac{\sigma}{e}}$$

Where, E = Young's Modulus
or, Modulus of elasticity
σ = Stress
e = Strain

Impact loads:- Impact loads are the loads that occurs suddenly and often causes stress on a component due to a dynamic event.

Stress :- The internal resistive force to deformation per unit ~~area~~ cross-sectional area is called as stress.



- Stress is denoted by σ (sigma).

$$\therefore \quad \sigma = \frac{P}{A}$$

Where, σ = stress
P = Load acting on the body
A = Cross-sectional area of the body

Mathematically, stress may be defined as the force per unit area.

- **S.I. Unit of stress**

$$\text{S.I. unit of stress} = \frac{\text{S.I. unit of force}}{\text{S.I. unit of area}}$$

$$\boxed{\text{S.I. unit of stress} = \frac{N}{m^2}}$$

- $\frac{N}{m^2}$ is called pascal and is denoted by Pa.

$$\therefore \boxed{\text{S.I. unit of stress} = Pa}$$

$$\boxed{1\ Pa = 1\ N/m^2}$$

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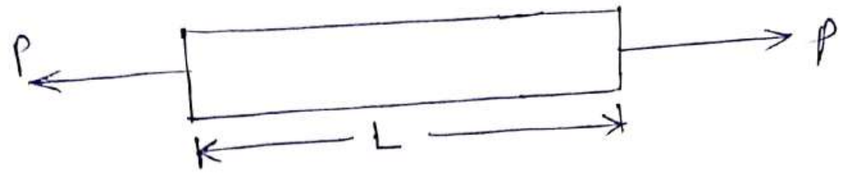

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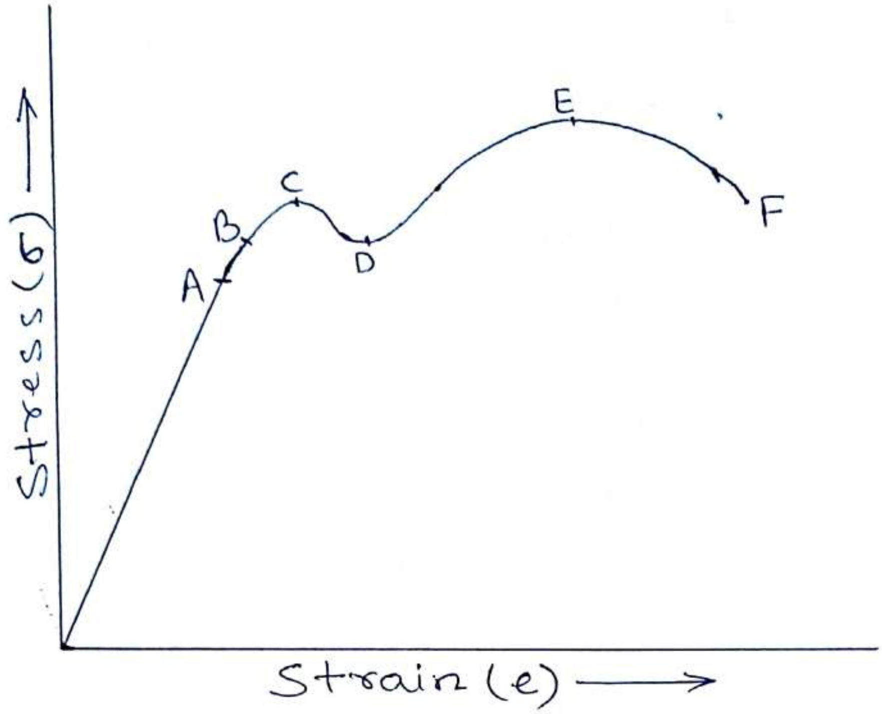
Stress-Strain Diagram (Ductile Materials)



The following are the salient features (Points) on the curve :-

A → proportional limit

B → Elastic Limit

C → Upper Yield point

D → Lower Yield point

E → Ultimate Load point

F → Breaking point

proportional limit (A) :-

It is defined as the maximum stress that can be applied to a material without producing a permanent deformation.

Elastic limit (B) :-

Elastic limit is the maximum stress a material can withstand before the permanent deformation.

Upper yield point :- The load at which the sudden drop occurs is known as the upper yield point.

Lower yield point :- It is a point at which minimum load required to maintain the plastic behavior of material.

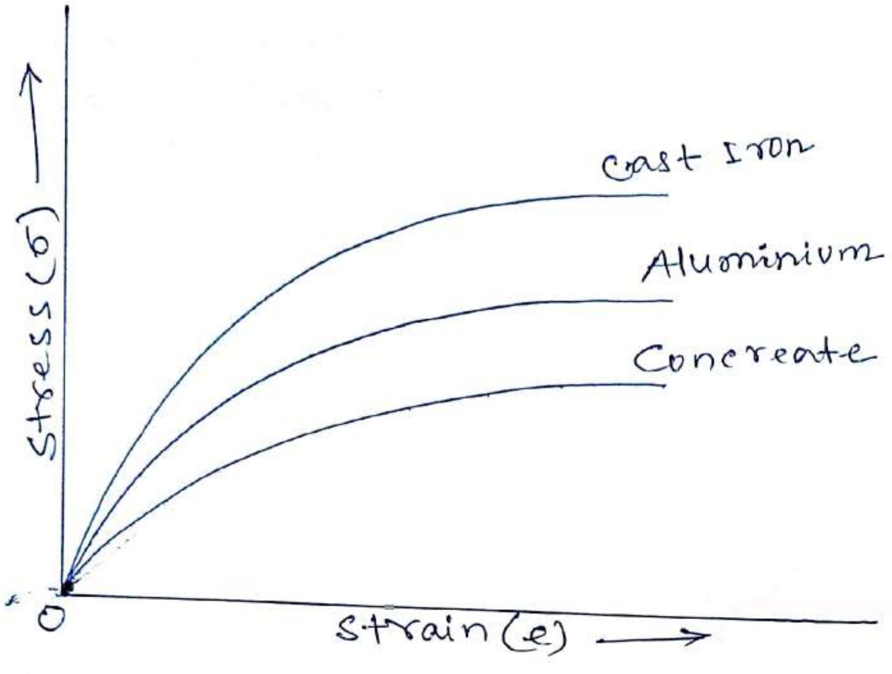
Yield point :- It is the maximum stress at which the material is ~~directly~~ deformed without considerable increase in load.

Ultimate load point :- It is defined as the highest load that the material can sustain before failure.

Or. It is defined as the maximum point on the stress - strain curve of the material.

Breaking point (F) :- It is defined as the the load point ~~in which the material~~ where fracture of material takes place.

Stress-Strain Diagram (Brittle Materials)



→ Brittle materials like Cast Iron, Aluminium, and concrete has very low proportional limit and do not show the yield point.

Linear strain:- The ratio of change in length to the original length of a body is called linear strain.

→ It is denoted by e.

$$\because \text{Linear strain} = \frac{\text{Change in length}}{\text{Original length}}$$

$$e = \frac{\delta L}{L}$$

where, e = Linear strain
δL = change in length
L = original length

→ Linear strain is also known as primary strain longitudinal strain.

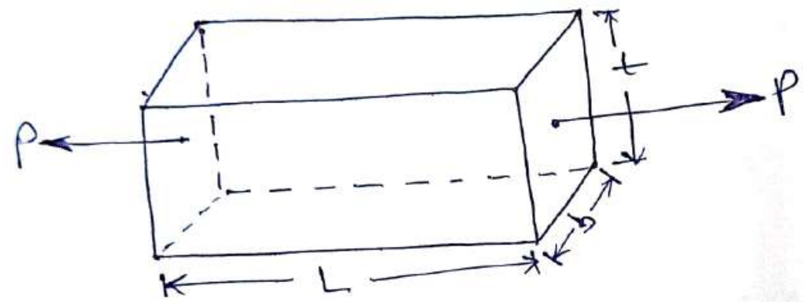
Lateral strain :- When a body is subjected to a load along the length, the ratio of change in breadth or width to the original breadth or width is called as lateral strain.

→ Lateral strain is also known as secondary or transverse strain.

Poisson's Ratio :- The ratio of lateral strain to linear strain is known as poisson's ratio.

→ poisson's ratio is denoted by μ.

→ μ varies from 0.1 to 0.5.



From figure,

$$\text{Linear strain, } e = \frac{\delta L}{L}$$

$$\text{Lateral strain} = \frac{-\delta b}{b} = \frac{-\delta t}{t}$$

Volumetric strain :- The ratio of change in volume to the original volume is known as volumetric strain.

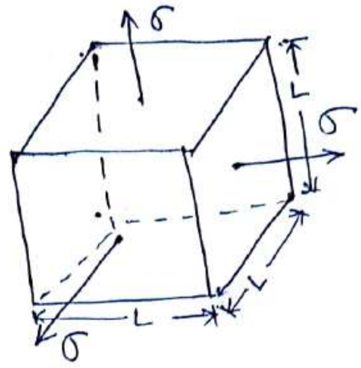
→ It is denoted by e_v.

$$\therefore \; e_v = \frac{\delta V}{V}$$

Where, e_v = volumetric strain
δV = change in volume
V = original volume.

Bulk Modulus :- It is defined as the ratio of normal stress to volumetric strain.

→ It is denoted by K.



$$\text{Bulk Modulus} = \frac{\text{Normal stress}}{\text{Volumetric strain}}$$

$$\boxed{K = \frac{\sigma}{e_v}}$$

→ S.I. of Bulk Modulus is N/m2 or pascal.

Modulus of Rigidity :- The ratio of shear stress and shear strain within the elastic limit is called as modulus of Rigidity.

→ It is denoted by G.

$$\text{Modulus of Rigidity} = \frac{\text{Shear stress}}{\text{Shear strain}}$$

$$\boxed{G = \frac{\tau}{\phi}}$$

where, G = Modulus of Rigidity
τ = Shear stress
ϕ = Shear strain

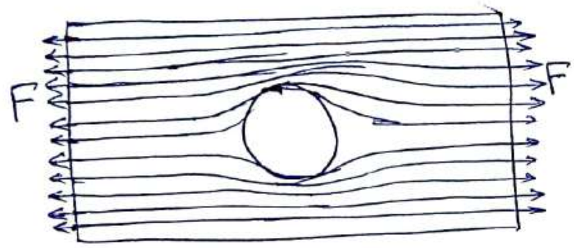
→ S.I. Unit of Modulus of Rigidity = N/m2

Fatigue :- The failure of a material under fluctuating and repeated loading is called fatigue.

Endurance Limit :-

Endurance limit is the stress level below which an infinite number of loading - cycles can be applied to a material without Causing fatigue failure.

Stress Concentration :- It is a location in an object where the stress is significantly greater than the surrounding region.



$$\text{Stress Concentration Factor } (K_t) = \frac{\text{Maximum Stress}}{\text{Average Stress}}$$

Factor of Safety :- The ratio of ultimate stress and the working stress for a material is called factor of safety.

Concept of Temperature Stresses :-

Temperature stresses and strains of uniform and composite sections

If the temperature of a body is lowered or raised its dimensions will decrease or increase correspondingly.
If these changes, however, are checked the stresses thus developed in the body are called temperature stresses and corresponding strains are called temperature strains.

Let, L = Length of a bar of uniform cross-section

t_1 = initial temperature of the bar

t_2 = final temperature of the bar,

α = Co-efficient of linear expansion

The extension in the bar due to rise in temperature will be -

$$\delta L = \alpha (t_2 - t_1) L$$

If this elongation in the bar is prevented by some external force or by fixing the bar ends, the temperature strain thus produced will be given by

$$\text{Temperature strain, } e_t = \frac{\text{Change in length}}{\text{original Length}} = \frac{\alpha (t_2 - t_1) L}{L}$$

Temperature strain, $e_t = \alpha(t_2 - t_1)$ [Compressive]

Now, Temperature stress developed, $\sigma_t = \alpha(t_2 - t_1) \times E$ [Compr

If, however, the temperature of the bar is lowered, the temperature strain and stress will be tensile in nature.

Q. A hollow circuler steel tube is 1000 mm in length. It is rigidly fixed at its ends at 80°c. If the temperature of the tube is lowered to 40°c, determine the magnitude and the nature of stresses developed.

Take $E = 2 \times 10^5$ N/mm² and $\alpha = 12 \times 10^{-6}$/°c.

<u>Solution</u>:- $L = 1000$ mm, $t_1 = 80°c$, $t_2 = 40°c$

Temperature stress, $\sigma_t = \alpha(t_2 - t_1)E$

$= 12 \times 10^{-6} \times (40 - 80) \times 2 \times 10^5$

$= -96$ N/mm² (compressive)

$\sigma_t = 96$ N/mm² (Tensile) Ans

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1. A bar 500 mm long and 22 mm in diameter is elongated by 1.2 mm under the effect of axial pull of 105 KN. Calculate the intensities of stress, strain and the modulus of elasticity of the bar.

Soln:–

Given, $L = 500mm$, $d = 22mm$, $\delta L = 1.2\,mm$, $P = 105\,KN = 105\times10^3 N$

We have to find:- (i) σ, (ii) e, (iii) E

(i) Stress, $\sigma = \frac{P}{A} = \frac{P}{\frac{\pi}{4}d^2} = \frac{105\times10^3}{\frac{\pi}{4}\times(22)^2} = 276.22\ N/mm^2$

(ii) Strain, $e = \frac{\delta L}{L} = \frac{1.2}{500} = 0.0024$

(iii) Modulus of elasticity, $E = \frac{\sigma}{e} = \frac{276.22}{0.0024} = 115091.67\ N/mm^2$

Q.2 A load of 6 KN is to be raised with the help of steel cable. Find the minimum diameter of steel cable if stress is not to exceed 110 N/mm².

Soln:- Given, load, $P = 6\ KN = 6\times10^3 N$

Stress, $\sigma = 110\ N/mm^2$

We have to find: Diameter, $d = ?$

We know that $\text{stress} = \frac{\text{Load}}{\text{Area}}$

or, $\sigma = \frac{P}{A}$

or, $110 = \frac{6\times10^3}{\frac{\pi}{4}\times d^2}$

or $d^2 = \frac{6\times10^3}{\frac{\pi}{4}\times110}$

or $d^2 = 54.54$

$\therefore d = 8.33\ mm$ Ans

Q.3 A rod having diameter 20 mm is subjected to an axial pull of 60 KN. The Length of the member is 2 meters. If $E = 2\times10^5\ N/mm^2$, find the deformation of the rod.

Solⁿ:- Given, $d = 20mm$, $P = 60KN = 60\times10^3\ N$
$L = 2m = 2\times10^3\ mm$, $E = 2\times10^5\ N/mm^2$

we have to find:- deformatio $= \delta L = ?$

we know that $\delta L = \frac{PL}{AE}$

$$= \frac{60\times10^3 \times 2\times10^3}{\frac{\pi}{4}\times(20)^2\times2\times10^5}$$

$$\delta L = 1.91\ mm \quad \underline{Ans}$$

Q.4 A 12 mm diameter M.S. bar when tested for Single Shear carries a load of 8.16 KN. Determine the shear stress induced.

Solⁿ:- Given, $d = 12mm$, $P = 8.16\ KN = 8.16\times10^3\ N$

We have to find:- Shear stress, $\tau = ?$

We know that, the relation for single shear as

$$\text{Shear stress, } \tau = \frac{\text{Shear load (Shear force)}}{\text{Area Subjected to Shear}}$$

$$\tau = \frac{P}{\frac{\pi}{4}d^2}$$

$$= \frac{8.16\times10^3}{\frac{\pi}{4}\times(12)^2}$$

$$\therefore \tau = 72.154\ N/mm^2 \quad \underline{Ans}$$

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Q.5 A 8 mm diameter circuler pin in double shear Carries a force of 10 KN. Determine the stress induced.

Solution:- Given, $d = 8mm$, $P = 10KN = 10\times10^3$ N

We have to find:- stress = τ = ?

We know that the double shear stress where area subjected to shear is

$$\text{Double Shear Stress, } \tau = \frac{\text{Shear force}}{\text{Area Subjected to Shear}}$$

$$\tau = \frac{P}{A}$$

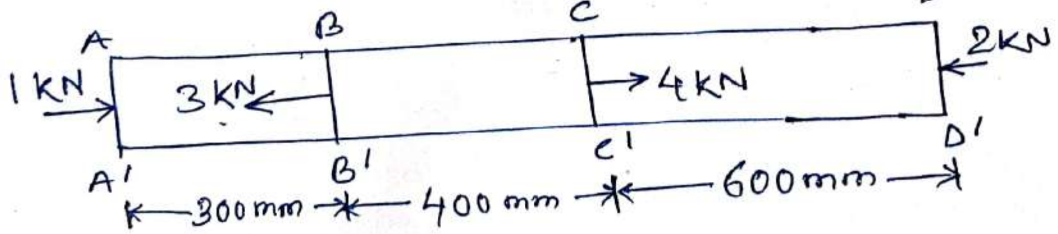
$$= \frac{10\times10^3}{2\times\frac{\pi}{4}\times(8)^2}$$

$$\therefore \tau = 99.43 \ N/mm^2 \ \text{Ans}$$

Deformation of a body Subjected to Axial loads

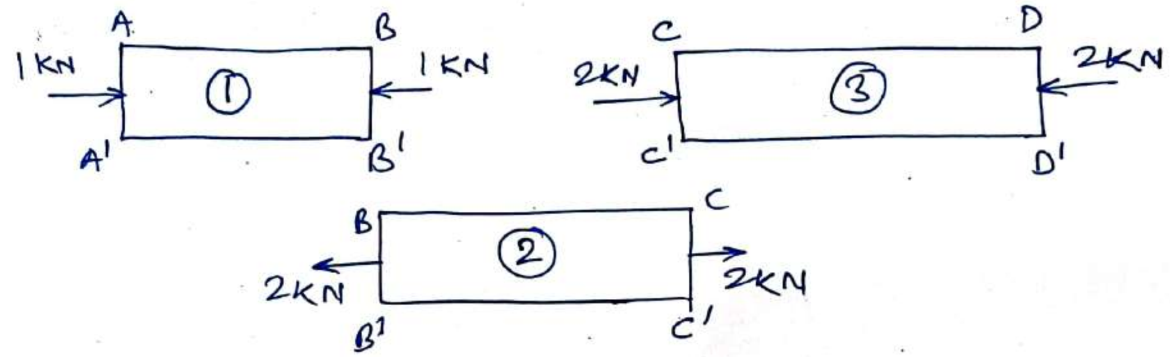
• principle of Superposition:- When a number of forces are acting on a body, the resulting strain will be the algebraic sum of strains of the individual sections.

Q. A bar of uniform cross sectional area 100 mm² is subjected to the forces as shown in fig. Calculate the change in the length of the bar. Take $E = 2\times10^5$ N/mm².



Soln :- Given: $A = 100\ mm^2$, $E = 2\times10^5\ N/mm^2$

The free body diagrams for the sections AB, BC & CD are shown.



Let δL_1, δL_2 and δL_3 are the change in the length of sections AB, BC and CD.

So, $\delta L_1 = -\frac{P_1 L_1}{AE} = \frac{-1000\times300}{100\times2\times10^5} = -0.015\ mm$

$\delta L_2 = \frac{P_2 L_2}{AE} = \frac{2000\times400}{100\times2\times10^5} = 0.04\ mm$

$\delta L_3 = -\frac{P_3 L_3}{AE} = \frac{-2000\times600}{100\times2\times10^5} = -0.06\ mm.$

∴ change in the length of the bar

$$\delta L = \delta L_1 + \delta L_2 + \delta L_3$$

$$\therefore\ \delta L = -0.015 + 0.04 - 0.06$$

$$\therefore\ \delta L = -0.035\ mm.$$

∵ Here, -ve sign shows the decrease in the length of the bar.

Q. In order to evaluate various mechanical properties, a steel specimen of 12.5 mm diameter and 62.5 mm gauge was tested in a standard tension test. Following observations were made during the test:

yield load = 40 KN, Maximum load = 71.5 KN,
Fracture load = 50.5 KN, Gauge length of fracture = 79.5 mm
Strain at load of 20 KN = 7.75×10^{-4}.

Determine (a) yield point stress, (b) ultimate tensile strength, (c) percentage elongation, (d) Modulus of elasticity (e) fracture stress

Soln:- $d = 12.5\,mm = 12.5 \times 10^{-3}\,m$

Original area of cross-section (A) $= \frac{\pi}{4} d^2$

$$\therefore A = \frac{\pi}{4} \times (12.5 \times 10^{-3})^2$$

$$A = 1.227 \times 10^{-4}\,m^2$$

(a) yield point stress, $\sigma_y = \frac{\text{Load at lower yield point}}{\text{original area}}$

$$= \frac{40 \times 10^3}{1.227 \times 10^{-4}} = 325.95 \times 10^6\,N/m^2$$

(b) Ultimate tensile strength $= \frac{\text{Maximum Load}}{\text{Original area}}$

$$= \frac{71.5 \times 10^3}{1.227 \times 10^{-4}}$$

$$= 582.6 \times 10^6\,N/m^2$$

(c) percentage elongation $= \frac{\text{length at fracture} - \text{original length}}{\text{original Length}} \times 100$

$$\text{percentage length} = \frac{79.5-62.5}{62.5} \times 100$$

$$= 27.2\%$$

(d) Modulus of elasticity (E)

$$\text{Stress at } 20\,KN = \frac{20\times10^{3}}{1.227\times10^{-4}} = 162.97\times10^{6}\ N/m^2$$

$$\text{Modulus of Elasticity, } E = \frac{\text{Stress at } 20\,KN}{\text{Strain at } 20\,KN}$$

$$= \frac{162.97\times10^{6}}{7.75\times10^{-4}}$$

$$= 2.1\times10^{11}\ N/m^2$$

(e) $$\text{Fracture stress} = \frac{\text{Fracture load}}{\text{Original area}}$$

$$= \frac{50.5\times10^{3}}{1.227\times10^{-4}}$$

$$= 411.5\times10^{6}\ N/m^2$$

Q. For a certain material, the modulus of elasticity is 2.8 times its bulk modulus. Calculate the poisson's ratio. Also Calculate the ratio of modulus of elasticity to modulus of rigidity.

Soln:- Given, $E = 2.8\,K$

We have to find:- (i) μ, (ii) E/G

We know that, $E = 3K(1-2\mu)$

$$\text{or, } 2.8K = 3K(1-2\mu)$$

$$\text{or, } 1-2\mu = \frac{2.8}{3} = 0.93$$

$$\text{or } 2\mu = 1-0.93 = 0.07$$

$$\text{or } \mu = \frac{0.07}{2} = 0.035 \quad \underline{\text{Ans}}$$

We know that $E = 2G(1+\mu)$

$$\text{or, } \frac{E}{G} = 2(1+\mu) = 2(1+0.035)$$

$$\text{or, } \frac{E}{G} = 2\times 1.035$$

$$\therefore \frac{E}{G} = 2.07 \quad \underline{\text{Ans}}$$

Q. A bar of 30 mm diameter is subjected to a pull of 60 KN. The measured extension on gauge length of 200 mm is 0.09 mm and the change in diameter is 0.0039 mm. Calculate the poisson's ratio and modulus of elasticity.

Soln :-

Given, $d = 30mm$, $P = 60KN = 60\times10^3 N$, $L = 200mm$,

$\delta L = 0.09mm$, $\delta d = 0.0039$.

We have to find :- μ and E.

$$\text{Linear strain}, e = \frac{\delta L}{L} = \frac{0.09}{200} = 0.00045$$

$$\text{Lateral Strain} = \frac{\delta d}{d} = \frac{0.0039}{30} = 0.00013$$

$$\text{poisson's ratio}, \mu = \frac{\text{Lateral Strain}}{\text{Linear strain}}$$

$$= \frac{0.00013}{0.00045}$$

$$\therefore \mu = 0.29$$

$$\text{We know}; E = \frac{\text{Stress}}{\text{Strain}} = \frac{\sigma}{e} = \frac{P/A}{\delta L/L} = \frac{PL}{A\delta L}$$

$$\text{or, } E = \frac{60\times10^3\times200}{\frac{\pi}{4}\times(30)^2\times0.09}$$

$$\therefore E = 1.88628\times10^5 \ N/mm^2$$ Ans

Q. For a given material, Young's Modulus is 1×10^5 N/mm^2 and modulus of rigidity 0.4×10^5 N/mm^2. Find the bulk modulus and lateral contraction of a round bar of 50 mm diameter and 2.5 m long, when stretched 2.5 mm. Take poisson's ratio as 0.25.

Soln:-

Given, $E = 1\times10^5$ N/mm^2, $G = 0.4\times10^5$ N/mm^2;

$d = 50$ mm, $L = 2.5$ m $= 2.5\times10^3$ mm,

$\delta L = 2.5$ mm, $\mu = 0.25$

We have to find:- K and δd.

We know that $E = 3K(1-2\mu)$

or, $1\times10^5 = 3\times K(1-2\times0.25) = 3K\times0.5 = 1.5K$

or $K = \frac{10^5}{1.5} = 0.67\times10^5$ N/mm^2

$\mu = \frac{\text{Lateral Strain}}{\text{Linear strain}}$

$0.25 = \frac{\text{Lateral Strain}}{\delta L/L} = \frac{\text{Lateral Strain}}{2.5/2.5\times10^3}$

or, Lateral strain $= 0.25\times\frac{2.5}{2.5\times103} = 0.00025$

or $\frac{\delta d}{d} = 0.00025$

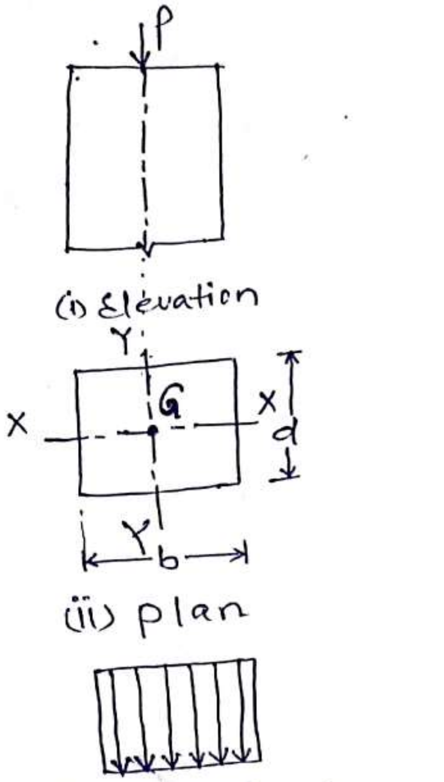
or $\frac{\delta d}{50} = 0.00025$

or $\delta d = 0.00025\times50$

$\therefore \delta d = 0.0125$ mm Ans

Combination of direct and bending stresses

* Concept of direct load



(i) Elevation

(ii) plan

(iii) Stress distribution diagram.

A Column of rectangular section having width 'b' and thickness 'd'.
An axial compressive load = P.
Load is acting exactly over the centroid G of the section.
The direct stress due to load 'P' of compressive nature whose intensity is uniform throughout the cross-section.

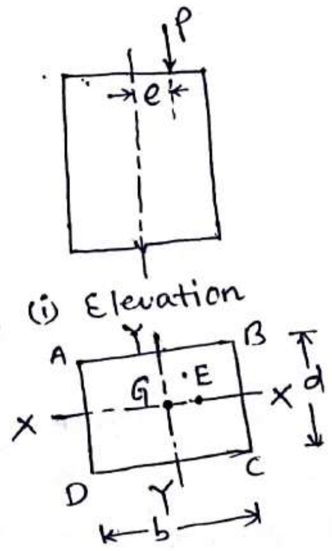
Stress at any point in the cross-section = P/A

→ Direct stress is denoted by σ_0.

Cross sectional area of column, $A = b \times d$

$$\boxed{\therefore \text{Direct stress}, \ \sigma_0 = \frac{P}{A} = \frac{P}{b \times d} \ (\text{compressive})}$$

* Concept of eccentric load



(i) Elevation

(ii) plan

→ The distance between the geometric axis of the body and the point of loading is called an eccentric limit or limit of eccentricity.
→ It is denoted by 'e'.
→ Axial load causes only direct stress whereas an eccentric load causes direct as well as bending stresses.

The load 'p' acts at point E in a plane bisecting the thickness i.e. on x-x axis i.e. it is eccentric with respect to Y-Y axis.

→ The distance GE is called eccentricity.

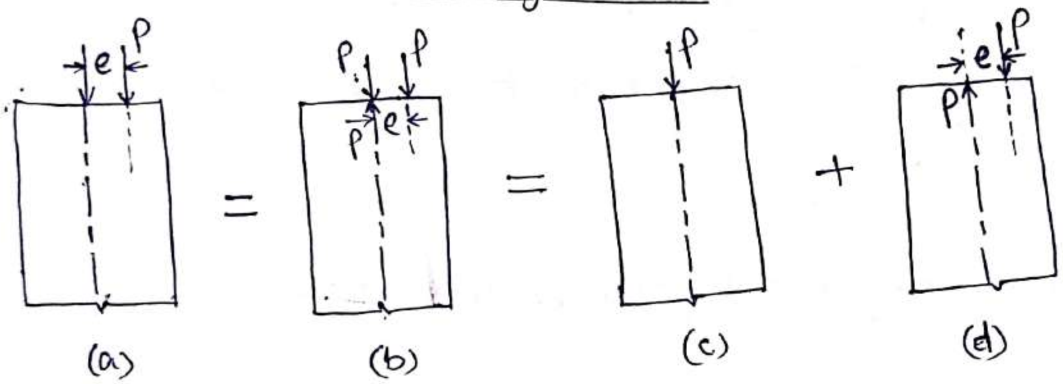
Combination of direct and bending stresses

Axial load:- A load whose line of action coincides with the axis of a member is called axial load.

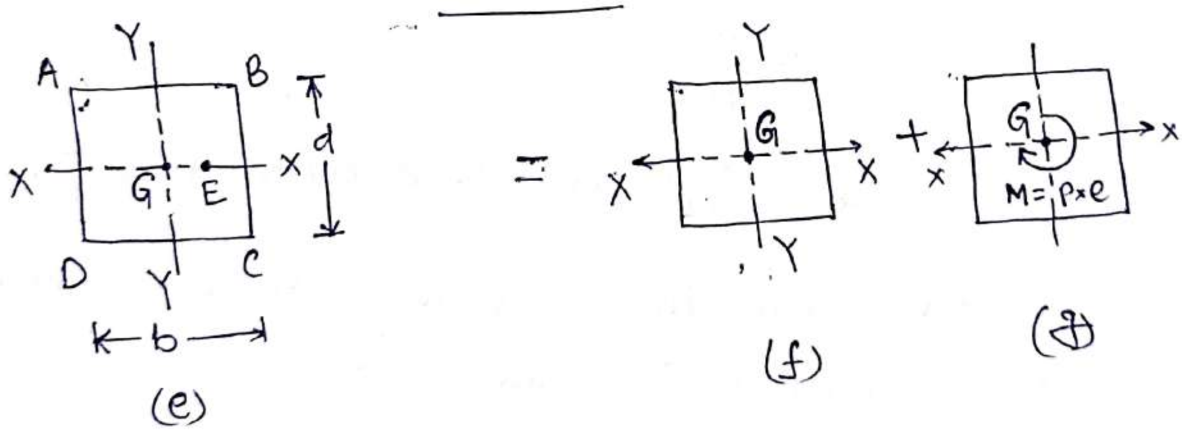
→ Axial load is also called direct load.

Eccentric load: A load whose line of action ~~acts~~ does not coincides with the axis of a member is called an eccentric load.

Direct Stress and bending stress



Elevation



Plan

Eccentric load at E = Direct load at G + Moment at G

→ The downward force P acting at the centroid G [Fig. (f)] will cause a direct stress.

→ The remaining two forces viz. a downward eccentric force P and an upward axial force P will from a clockwise couple whose moment at the centroid G will be $M = P \times e$.

This moment acting at the centroid G will cause a <u>bending stress</u>.

→ A body subjected to an eccentric loading causes direct as well as bending stress.

Maximum and Minimum Stress

Direct stress due to eccentric load P,

$$\sigma_0 = \frac{P}{A}$$

Bending stress (σ_b) due to eccentric load P:

We know that the bending stress equation i.e. flexural formula.

$$\frac{M}{I} = \frac{\sigma_b}{y}$$

$$\sigma_b = \frac{M}{I} \times y = \frac{M}{(I/y)} = \frac{M}{Z}$$

$$\boxed{\sigma_b = \frac{M}{Z}}$$

Where, I = M.I. of the column section about the neutral axis $Y-Y = db^3/12$

y = Distance of the layer from the neutral axis $Y-Y$.

σ_b = Bending stress in a layer at a distance y from the neutral axis

Z = Section modulus $= \frac{I}{y}$

The resultant stress at a distance y from the neutral axis $Y-Y$ is given by,

σ_y = Direct stress $\pm$ Bending stress

$$\sigma_y = \sigma_0 \pm \sigma_b$$

Maximum stress, $\sigma_{max} = \sigma_0 + \sigma_b$

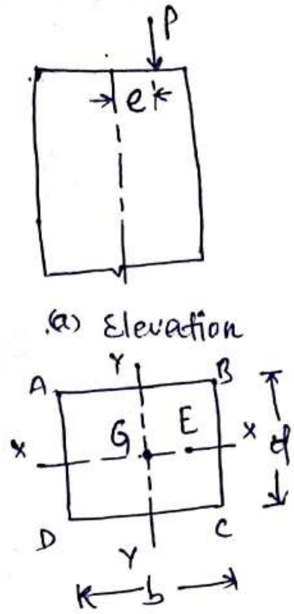
Minimum stress, $\sigma_{min} = \sigma_0 - \sigma_b$

Stress distribution Diagram

σ_{max} = Direct stress + Bending stress

$$\sigma_{max} = \sigma_0 + \sigma_b$$

$$\boxed{\sigma_{max} = \frac{P}{A} + \frac{M}{Z} \text{ (Compressive)}}$$



(a) Elevation

(b) plan

σ_{min} = Direct stress − Bending stress

$$\sigma_{min} = \sigma_0 - \sigma_b$$

$$\boxed{\sigma_{min} = \frac{P}{A} - \frac{M}{Z}}$$

⇒ Nature of σ_{min} depends upon the magnitudes of σ_0 and σ_b.

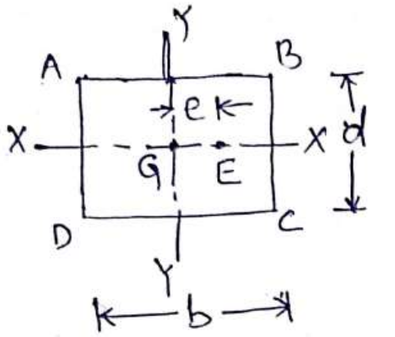
⇒ For calculating the resultant stress due to eccentric loading as shown in above Fig., three cases are possible.

(i) If $\sigma_0 > \sigma_b$, the stress throughout the section will be of the same nature i.e. compressive.

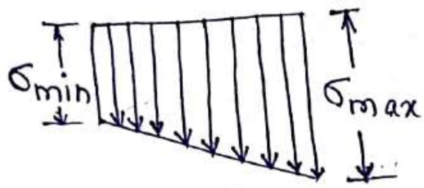
(ii) If $\sigma_0 = \sigma_b$, then also the stress throughout the section will be of the same nature i.e. compressive.

(iii) If $\sigma_0 < \sigma_b$, the stress will be partly tensile and partly compressive.

The stress distribution diagrams for the above three cases are as shown in Fig. below.



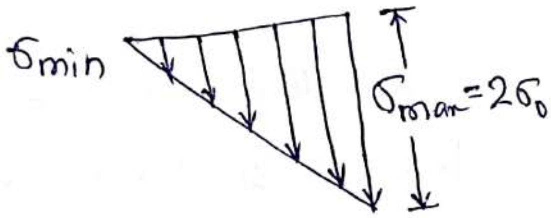
Eccentric compressive load at 'E'



Stress distribution totally compressive

$\sigma_0 > \sigma_b$

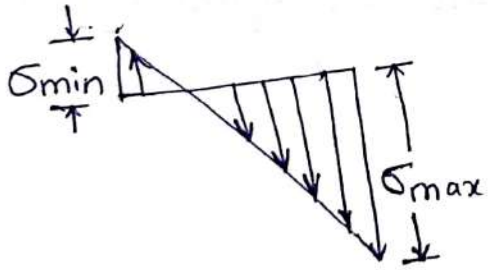
(i)



Stress distribution totally compressive

$\sigma_0 = \sigma_b$

(ii)



Stress distribution partly tensile and partly compressive

$\sigma_0 < \sigma_b$

Condition for no tension in the section

When an eccentric compressive load acts on a column, it produces direct as well as bending stress.

→ If $\sigma_0 > \sigma_b$, the resultant stress is compressive.

→ If $\sigma_0 = \sigma_b$, the minimum stress is zero and the maximum stress is $2\sigma_0$ and the stress distribution is compressive.

→ If $\sigma_0 < \sigma_b$, the stress is partly compressive and partly tensile.

→ A small tensile stress at the base of a structure may develop tension cracks. Hence, for no-tension condition, direct stress should be greater than or equal to bending stress.

$$\sigma_0 \geq \sigma_b$$

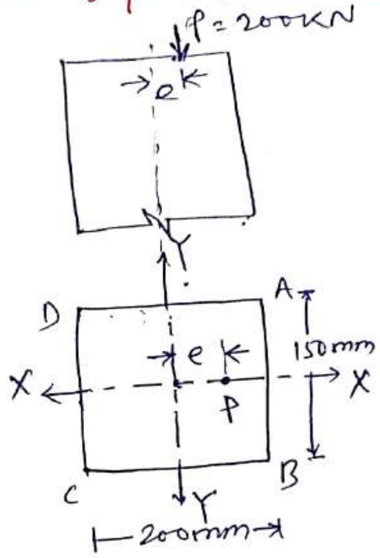
$$\frac{P}{A} \geq \frac{M}{Z}$$

$$\frac{P}{A} \geq \frac{P.e}{Z}$$

$$\boxed{e \leq \frac{Z}{A}}$$

Hence, for no-tension condition, eccentricity should be less than Z/A, or maximum value equal to Z/A.

Q. A Column section 200 mm wide and 150 mm thick is subjected to a load of 200 KN at an eccentricity of 20 mm in a plane - bisecting the thickness. Find the maximum and minimum intensities of stress in the section.



Solⁿ: Given:- $b = 200\ mm$,

$d = 150\ mm$,

$P = 200\ kN = 200 \times 10^3\ N$

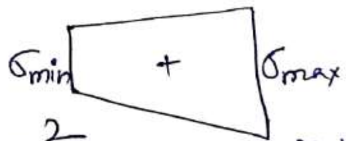
$e = 20\ mm$

Find:- σ_{max} & σ_{min}.

Area of section, $A = b \times d$

$= 200 \times 150$

$= 3 \times 10^4\ mm^2$



Stress distribution

Direct stress, $\sigma_0 = \frac{P}{A}$

$= \frac{200 \times 10^3}{3 \times 10^4}$

$= 6.67\ N/mm^2$ (Compressive)

Bending stress, $\sigma_b = \frac{M}{Z_{YY}}$

Here axis of bending is YY - axis.
The load is eccentric w.r.t. YY axis.

$Z_{YY} = \frac{d b^2}{6}$

$$\sigma_b = \frac{P \times e}{db^2/6}$$

$$= \frac{200 \times 10^3 \times 20}{150 \times 200^2/6}$$

$$\sigma_b = 4\ N/mm^2$$

$\sigma_{max} = \sigma_0 + \sigma_b = 6.67 + 4 = 10.67\ N/mm^2$ (compressive)

$\sigma_{min} = \sigma_0 - \sigma_b = 6.67 - 4 = 2.67\ N/mm^2$ (compressive)

Ans

CH-04: Combined Stress

* Important Formulae

(i) Direct stress, $\sigma_0 = \frac{P}{A}$

(ii) Bending stress, $\sigma_b = \frac{M}{Z_{YY}}$ [if the load is eccentric w.r.t. YY axis]

Bending stress, $\sigma_b = \frac{M}{Z_{XX}}$ [if the load is eccentric w.r.t. XX axis]

(iii) $\sigma_{max} = \sigma_0 + \sigma_b$ (compressive)

$\sigma_{min} = \sigma_0 - \sigma_b$ (compressive if $\sigma_0 > \sigma_b$ and tensile if $\sigma_0 < \sigma_b$)

(iv) If $\sigma_0 > \sigma_b$, the stress distribution diagram is totally compressive.

If $\sigma_0 = \sigma_b$, the stress distribution diagram is totally compressive.

$\sigma_{max} = 2\sigma_0$ and $\sigma_{min} = 0$

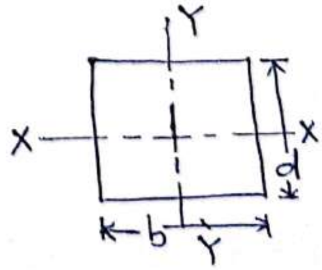
If $\sigma_0 < \sigma_b$, the stress distribution diagram is partly tensile and partly compressive.

(v) values of 'Z' (Section modulus) for different sections:

1. For a column of rectangular sections of width 'b' and depth 'd'.

$$Z_{XX} = \frac{I_{YY}}{Y_{max}} = \frac{bd^3/12}{d/2} = \frac{bd^2}{6}$$

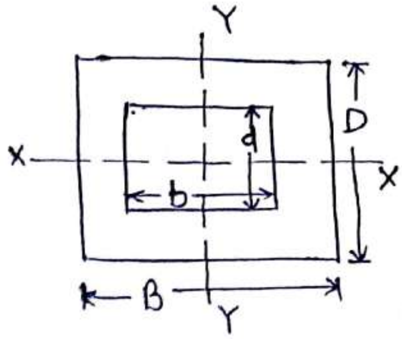
$$Z_{YY} = \frac{I_{YY}}{Y_{max}} = \frac{db^3/12}{b/2} = \frac{db^2}{6}$$



2. For a column of hollow rectangular section having external dimensions B and D and internal dimensions b and d.

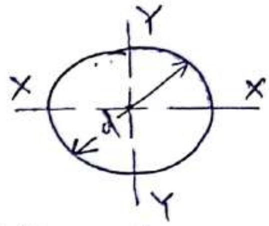
$$Z_{xx} = \frac{I_{xx}}{Y_{max}} = \frac{\frac{BD^3 - bd^3}{12}}{\frac{D}{2}} = \frac{BD^3 - bd^3}{6D}$$

$$Z_{YY} = \frac{I_{YY}}{Y_{max}} = \frac{\frac{DB^3 - db^3}{12}}{\frac{B}{2}} = \frac{DB^3 - db^3}{6B}$$



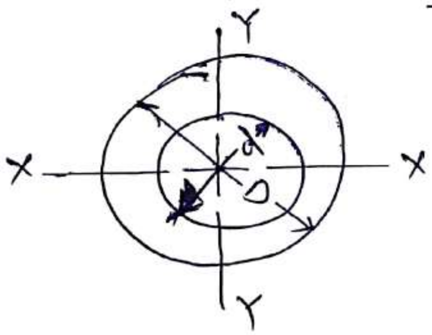
3. For a column of solid circular section of diameter 'd'.

$$Z = Z_{xx} = Z_{YY} = \frac{I}{Y_{max}} = \frac{\frac{\pi d^4}{64}}{d/2} = \frac{\pi d^3}{32}$$



4. For a column of circular section of external diameter 'D' and internal diameter 'd'.

$$Z = Z_{xx} = Z_{YY} = \frac{I}{Y_{max}} = \frac{\frac{\pi D^4}{64} - \frac{\pi d^4}{64}}{\frac{D}{2}} = \frac{\pi}{32}\left(\frac{D^4 - d^4}{D}\right)$$



Q.1 A Rectangular column is 200 mm wide and 100 mm ~~thick wide~~ thick. It is subjected to a load of 180 KN at an eccentricity of 100 mm in the plane bisecting the thickness.
Draw the combined stress distribution diagram showing their values.

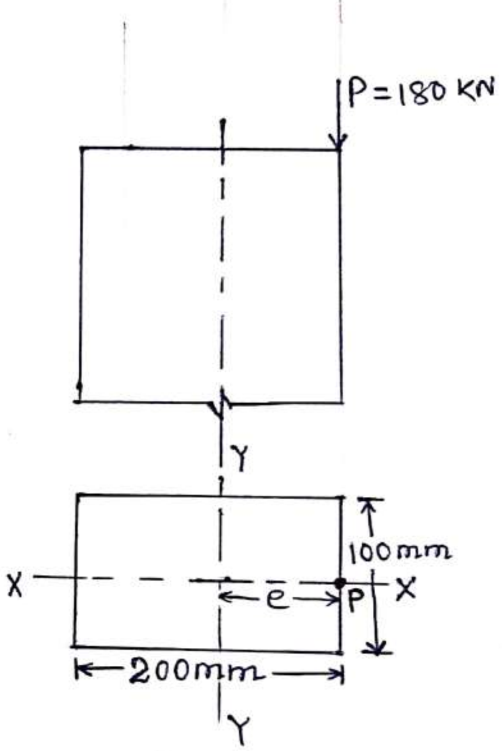
Soln :- Given:- wide, $b = 200\,mm$

Thick, $d = 100\,mm$

Load, $P = 180\,KN = 180 \times 10^3\,N$

Eccentricity, $e = 100\,mm$

Find: σ_{max} and σ_{min}



Cross-sectional area of

Column, $A = b \times d$

$= 200 \times 100$

$A = 2 \times 10^4\,mm^2$

Direct stress, $\sigma_0 = \frac{P}{A}$

$$\sigma_0 = \frac{180 \times 10^3}{2 \times 10^4}$$

$$\boxed{\sigma_0 = 9\,N/mm^2 \text{ (Compressive)}}$$

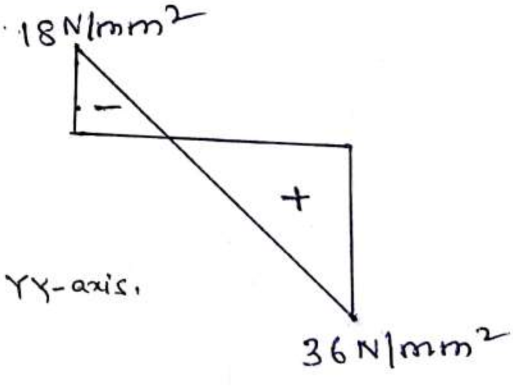


Fig: stress distribution diagram.

∴ The load is eccentric with respect to YY-axis.

Bending stress, $\sigma_b = \frac{M}{Z_{YY}}$

$$\sigma_b = \frac{P \times e}{db^2/6} = \frac{6Pe}{db^2}$$

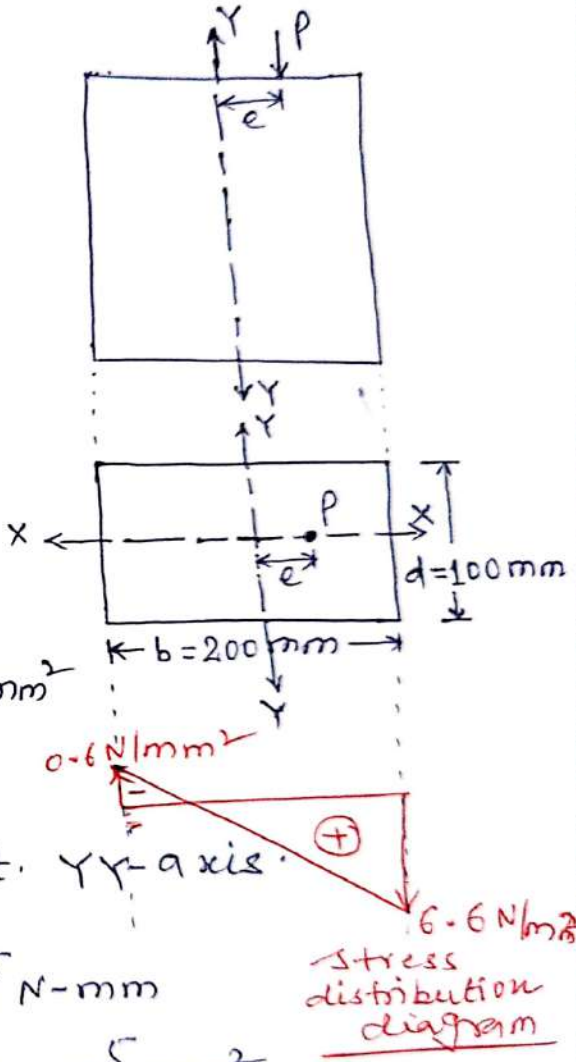
$$\sigma_b = \frac{6 \times 180 \times 10^3 \times 100}{100 \times (200)^2} = 27\,N/mm^2$$

$\sigma_{max} = \sigma_0 + \sigma_b = 9 + 27 = 36\,N/mm^2$ (Compressive)

$\sigma_{min} = \sigma_0 - \sigma_b = 9 - 27 = -18\,N/mm^2$ (Tensile)

Q.2 A short column 200 mm × 100 mm is subjected to an eccentric load of 60 kN at an eccentricity of 40 mm in the plane bisecting the 100 mm side. Find the maximum and minimum intensities of stresses at the base.

Solution:—



Given:- width, $b = 200$ mm

depth, $d = 100$ mm

Load, $P = 60\,kN = 60 \times 10^3$ N

eccentricity, $e = 40$ mm

Find:- σ_{max} & σ_{min}.

We know that

$\sigma_{max} = \sigma_0 + \sigma_b$

$\sigma_{min} = \sigma_0 - \sigma_b$

σ_0 = direct stress $= \frac{P}{A}$

Area, $A = b \times d = 200 \times 100 = 2 \times 10^4\ mm^2$

$\sigma_0 = \frac{P}{A} = \frac{60 \times 10^3}{2 \times 10^4} = 3\ N/mm^2$

∴ The load is eccentric w.r.t. YY-axis.

$\sigma_b = \frac{M}{Z_{YY}}$

$M = P \times e = 60 \times 10^3 \times 40 = 24 \times 10^5$ N-mm

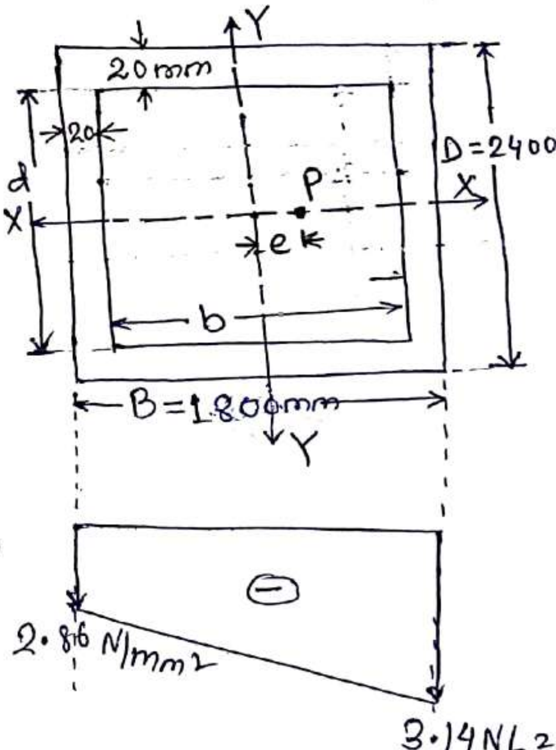
$Z_{YY} = \frac{db^2}{6} = \frac{100 \times (200)^2}{6} = 6.67 \times 10^5\ mm^3$

$\sigma_b = \frac{M}{Z_{YY}} = \frac{24 \times 10^5}{6.67 \times 10^5} = 3.6\ N/mm^2$

$\sigma_{max} = \sigma_0 + \sigma_b = 3 + 3.6 = 6.6\ N/mm^2$ (compressive)

$\sigma_{min} = \sigma_0 - \sigma_b = 3 - 3.6 = -0.6\ N/mm^2$ (Tensile)

Q.3 A short column of hollow rectangular cross-section has external dimensions 1800mm × 2400mm and is 20 mm thick. It carries a vertical load of 500KN at an eccentricity of 30mm from the geometric axis of the section bisecting the longer side. Find σ_{max} and σ_{min}.



Stress distribution diagram

Solution: –

Given: Outer dimensions:

$B = 1800mm, \; D = 2400 mm$

Inner dimensions:

$b = 1800 - 20 - 20 = 1760 mm$

$d = 2400 - 20 - 20 = 2360 mm$

Cross-sectional area of the column $\Rightarrow A = BD - bd$

$= 1800 \times 2400 - 1760 \times 2360$

$A = 166400 \; mm^2$

Eccentricity, $e = 30 \; mm$

Load, $P = 500 KN = 500 \times 10^3 N$

Find:- σ_{max} & σ_{min}.

The load is eccentric w.r.t. YY axis.

Section modulus, $Z_{YY} = \frac{I_{YY}}{y_{max}}$

$$Z_{YY} = \frac{DB^3 - db^3}{6B}$$

$$Z_{YY} = \frac{2400 \times 1800^3 - 2360 \times 1760^3}{6 \times 1800}$$

$$Z_{YY} = 104685985.2 \; mm^3$$

direct stress, $\sigma_0 = \frac{P}{A}$

$$\sigma_0 = \frac{500 \times 10^3}{166400} = \underline{3\ N/mm^2} \text{ (compressive)}$$

Bending stress, $\sigma_b = \frac{M}{Z_{YY}} = \frac{P \cdot e}{Z_{YY}}$

$$= \frac{500 \times 10^3 \times 30}{104685985.2}$$

$$\boxed{\sigma_b = 0.14\ N/mm^2}$$

$\sigma_{max} = \sigma_0 + \sigma_b = 3 + 0.14 = 3.14\ N/mm^2$ (compressive)

$\sigma_{min} = \sigma_0 - \sigma_b = 3 - 0.14 = 2.86\ N/mm^2$ (compressive)

Q.4 A hollow column of rectangular section 600 mm × 300 mm overall and 500 mm × 250 mm internally carries a load of 15 kN which is of the geometric axis by 100 mm in the vertical plane bisecting the thickness i.e. 300 mm side. Calculate the extreme intensities of stress induced in the section.

Solution:-

Given: Outer dimensions:

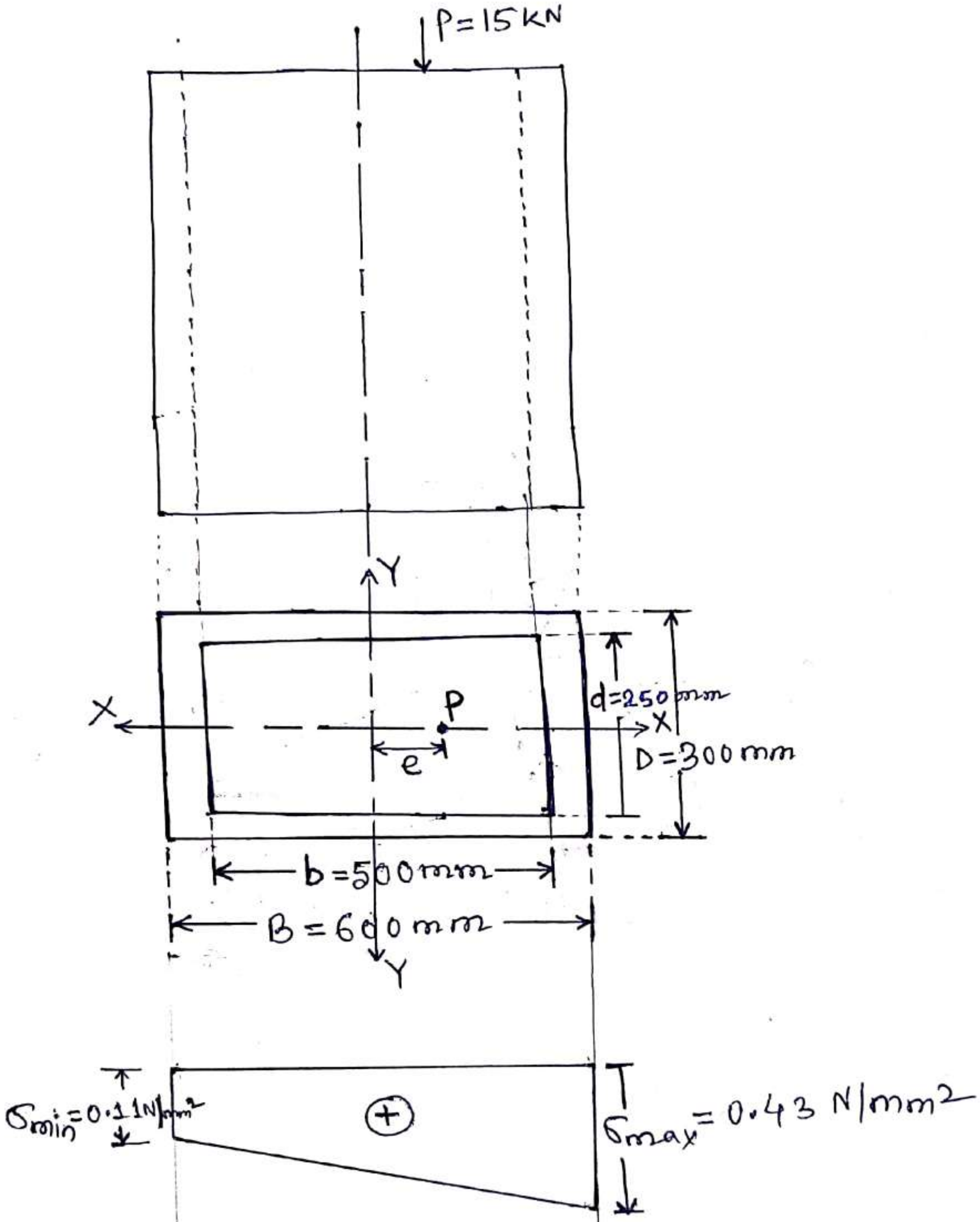
width, $B = 600$ mm

depth, $D = 300$ mm

Internal dimensions:-

width, $b = 500$ mm

depth, $d = 250$ mm



Stress Distribution Diagram

Load, $P = 15\,kN = 15 \times 10^3\,N$

Eccentricity, $e = 100\,mm$.

Find:- σ_{max} and σ_{min}.

We know that

$\sigma_{max} = \sigma_0 + \sigma_b$

$\sigma_{min} = \sigma_0 - \sigma_b$

Direct stress, $\sigma_0 = \frac{P}{A}$

Area of hollow section, $A = BD - bd$

$A = 600 \times 300 - 500 \times 250$

$\boxed{A = 55000\ mm^2}$

$\sigma_0 = \frac{15 \times 10^3}{55000} = \underline{0.27\ N/mm^2}$ (compressive)

The load is eccentric with respect to YY-axis.

$\sigma_b = \frac{M}{Z_{YY}}$

$M = P \times e = 15 \times 10^3 \times 100 = 15 \times 10^5\ N\text{-}mm$

$Z_{YY} = \frac{DB^3 - db^3}{6B}$

$= \frac{300 \times (600)^3 - 250 \times (500)^3}{6 \times 600}$

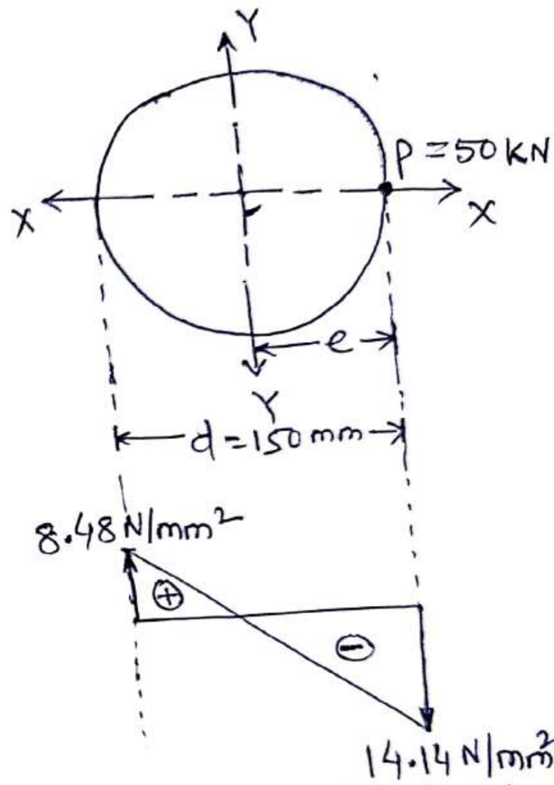
$Z_{YY} = 931944.44\ mm^3$

Bending stress, $\sigma_b = \frac{M}{Z_{YY}} = \frac{15 \times 10^5}{931944.44} = \underline{0.16\ N/mm^2}$

$\sigma_{max} = \sigma_0 + \sigma_b = 0.27 + 0.16 = 0.43\ N/mm^2$ (compressive)

$\sigma_{min} = \sigma_0 - \sigma_b = 0.27 - 0.16 = 0.11\ N/mm^2$ (compressive)

Q5. A solid circular column of diameter 150 mm carries a vertical load of 50 KN at outer edge of the column. Calculate σ_{max} and σ_{min}.



Stress Distribution Diagram

Solution:-

Given: diameter, $d = 150$ mm

Load, $P = 50$ KN $= 50\times10^3$ N

Eccentricity, $e = \frac{d}{2} = \frac{150}{2} = 75$ mm

Find: σ_{max} & σ_{min}.

We know that

$$\sigma_{max} = \sigma_0 + \sigma_b$$

$$\sigma_{min} = \sigma_0 - \sigma_b$$

Direct stress, $\sigma_0 = \frac{P}{A}$

Area, $A = \frac{\pi}{4}d^2 = \frac{22}{7\times4}(150)^2$

$$\boxed{A = 17678.57\ mm^2}$$

$$\sigma_0 = \frac{P}{A} = \frac{50\times10^3}{17678.57} = 2.83\ N/mm^2\ (\text{compressive})$$

∴ The load is eccentric w.r.t. diameter.

$$\sigma_b = \frac{M}{Z}$$

$$M = P\times e = 50\times10^3\times75 = 375\times10^4\ N\text{-}mm$$

$$Z = \frac{\pi}{32}d^3 = \frac{22}{7}\times\frac{1}{32}(150)^3 = 331473.21\ mm^3$$

$$\sigma_b = \frac{M}{Z} = \frac{375\times10^4}{331473.21} = 11.31\ N/mm^2$$

$$\sigma_{max} = \sigma_0 + \sigma_b = 2.83 + 11.31 = 14.14\ N/mm^2\ (\text{Compressive})$$

$$\sigma_{min} = \sigma_0 - \sigma_b = 2.83 - 11.31 = -8.48\ N/mm^2\ (\text{Tensile})$$

Q6. A rectangular column 300mm wide and 200 mm thick carries an axial load of 180 KN and a clockwise moment of 2.8 KN-m in the plane bisecting 200mm side. calculate the resultant stresses induced at the base.

Solution:-

Given: wide, $b = 300\,mm$

Thick, $d = 200\,mm$

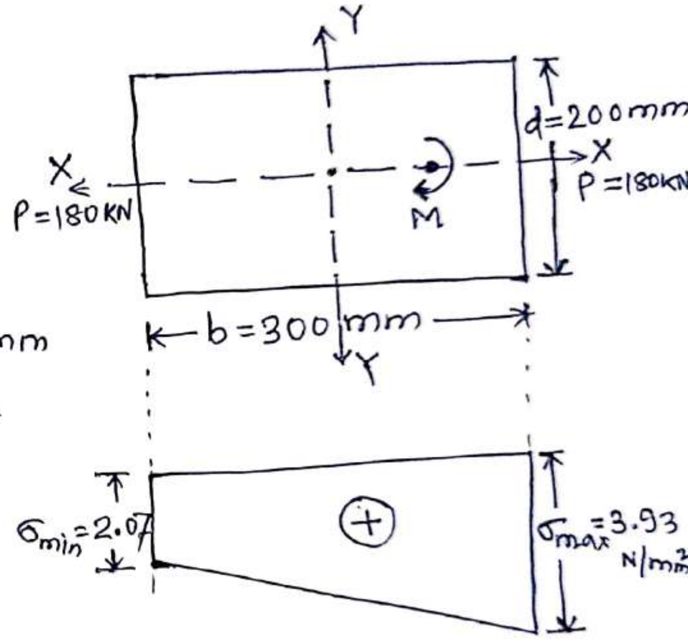
Load, $P = 180\,KN = 180\times10^3\,N$

Moment, $M = 2.8\,KN\text{-}m$

$= 2.8\times10^3\,N\text{-}m$

$= 2.8\times10^3\times10^3\,N\text{-}mm$

$= 2.8\times10^6\,N\text{-}mm$



Stress distribution diagram

Find: σ_{max} and σ_{min}.

We know that

$\sigma_{max} = \sigma_0 + \sigma_b$

$\sigma_{min} = \sigma_0 - \sigma_b$

σ_0, direct stress $= \frac{P}{A}$

A, Area $= b\times d$

$= 300\times200$

$= 6\times10^4\,mm^2$

$\sigma_0 = \frac{P}{A} = \frac{180\times10^3}{6\times10^4}$

$\therefore \boxed{\sigma_0 = 3\,N/mm^2\text{(compressive)}}$

The load is eccentric with respect to YY-axis.

σ_b, bending stress $= \frac{M}{Z_{YY}}$

$$Z_{YY} = \frac{db^2}{6}$$

$$\therefore \sigma_b = \frac{M}{Z_{YY}} = \frac{M}{\frac{db^2}{6}} = \frac{6M}{db^2} = \frac{6 \times 2.8 \times 10^6}{200 \times (300)^2} = \underline{0.93\ N/mm^2}$$

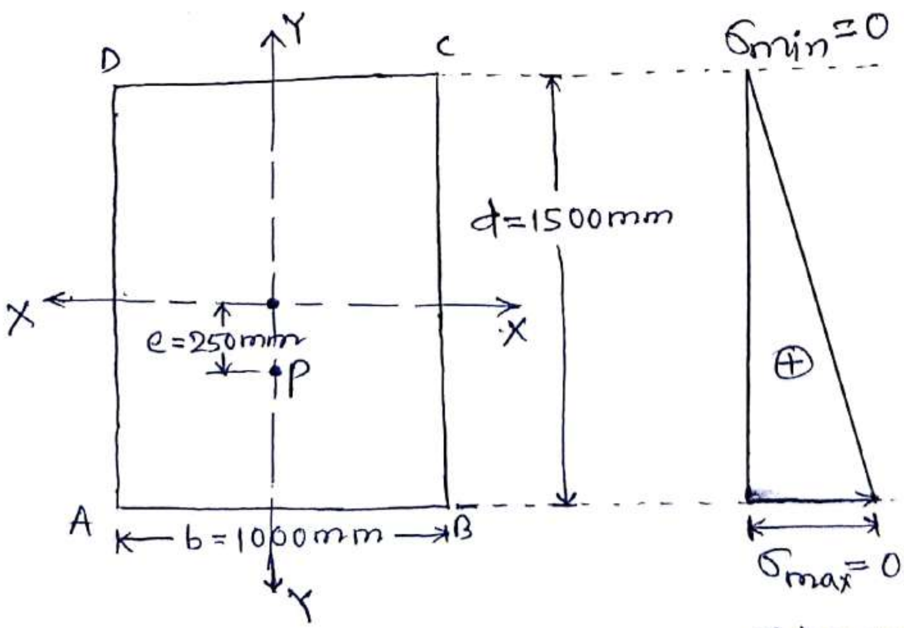
$$\sigma_{max} = \sigma_0 + \sigma_b = 3 + 0.93 = 3.93\ N/mm^2 \text{ (compressive)}$$

$$\sigma_{min} = \sigma_0 - \sigma_b = 3 - 0.93 = 2.07\ N/mm^2 \text{ (compressive)}$$

Ans

Q7. A rectangular pier 1000 mm × 1500 mm is subjected to a compressive load of 500 KN with an eccentricity of 250 mm along the axis bisecting 1000 mm side. Find the resultant stress intensities at the base cross-section of the pier.

Solution:-



Stress distribution diagram.

Given: width, $b = 1000\,mm$

breadth, $d = 1500\,mm$

Load, $P = 500\,KN = 500 \times 10^3\,N$

Eccentricity, $e = 250\,mm$

Find: σ_{max} and σ_{min}.

We know that

$$\sigma_{max} = \sigma_0 + \sigma_b$$

$$\sigma_{min} = \sigma_0 - \sigma_b$$

Direct stress, $\sigma_0 = \frac{P}{A}$

Area, $A = b \times d$.

$= 1000 \times 1500 = 15 \times 10^5\,mm^2$

$$\sigma_0 = \frac{P}{A} = \frac{500 \times 10^3}{15 \times 10^5} = \frac{1}{3} = 0.33\,N/mm^2$$

$$\boxed{\sigma_0 = 0.33\,N/mm^2}$$

The load is eccentric with respect to x-x axis.

$$\sigma_b = \frac{M}{Z_{xx}}$$

$M = P \times e = 500 \times 10^3 \times 250 = 125 \times 10^6\,N\text{-}mm$

$$Z_{xx} = \frac{bd^2}{6} = \frac{1000 \times (1500)^2}{6} = 375 \times 10^6\,mm^3$$

$$\sigma_b = \frac{M}{Z_{xx}} = \frac{125 \times 10^6}{375 \times 10^6} = \frac{1}{3} = 0.33\,N/mm^2$$

$\sigma_{max} = \sigma_0 + \sigma_b = 0.33 + 0.33 = 0.66\ N/mm^2$ (compressive on face AB)

$\sigma_{min} = \sigma_0 - \sigma_b = 0.33 - 0.33 = 0$ (on face CD)

Q8. A hollow circular column having external and internal diameters of 40 cm and 30 cm respectively, carries a vertical load of 150 KN at the outer edge of the column. Calculate the maximum and minimum intensities of stress in the section.

Solution:-

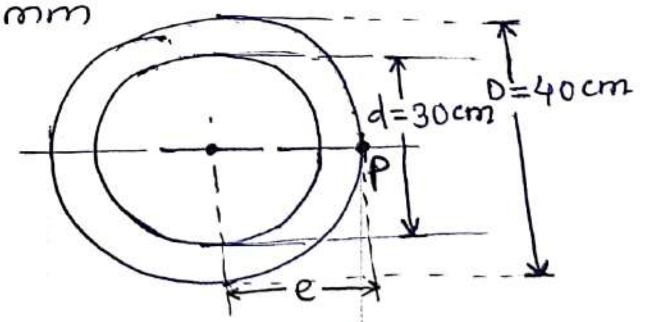
Given:

External diameter, $D = 40\ cm = 400\ mm$

Internal diameter, $d = 30\ cm = 300\ mm$

Load, $P = 150\ KN = 150 \times 10^3\ N$

Eccentricity, $e = \frac{D}{2} = \frac{400}{2} = 200\ mm$



Find: σ_{max} and σ_{min}

we know that

$\sigma_{max} = \sigma_0 + \sigma_b$

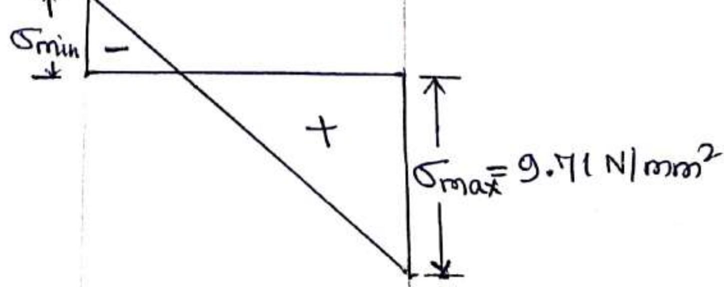
$\sigma_{min} = \sigma_0 - \sigma_b$

Direct stress, $\sigma_0 = \frac{P}{A}$

Area, $A = \frac{\pi}{4}(D^2 - d^2)$

$= \frac{\pi}{4}(400^2 - 300^2)$

$= 54977.87\ mm^2$



Stress Distribution Diagram

$$\sigma_0 = \frac{P}{A} = \frac{150\times10^3}{54977.87} = \underline{2.73\ N/mm^2}\ \text{(compressive)}$$

Bending stress, $\sigma_b = \frac{M}{Z}$

$$M = P\times e = 150\times10^3\times200 = 3\times10^7\ N\text{-}mm$$

$$Z = \frac{\pi}{32}\left[\frac{D^4-d^4}{D}\right]$$

$$= \frac{\pi}{32}\left[\frac{400^4-300^4}{400}\right]$$

$$Z = 4295146.21\ mm^3$$

$$\sigma_b = \frac{M}{Z} = \frac{3\times10^7}{4295146.21}$$

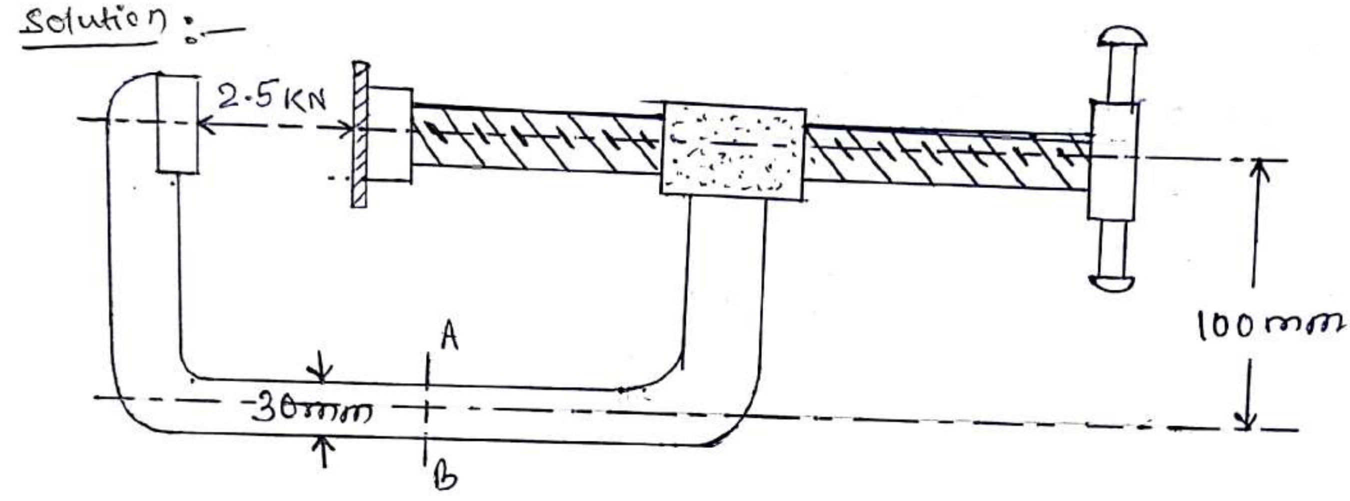
$$\boxed{\sigma_b = 6.98\ N/mm^2}$$

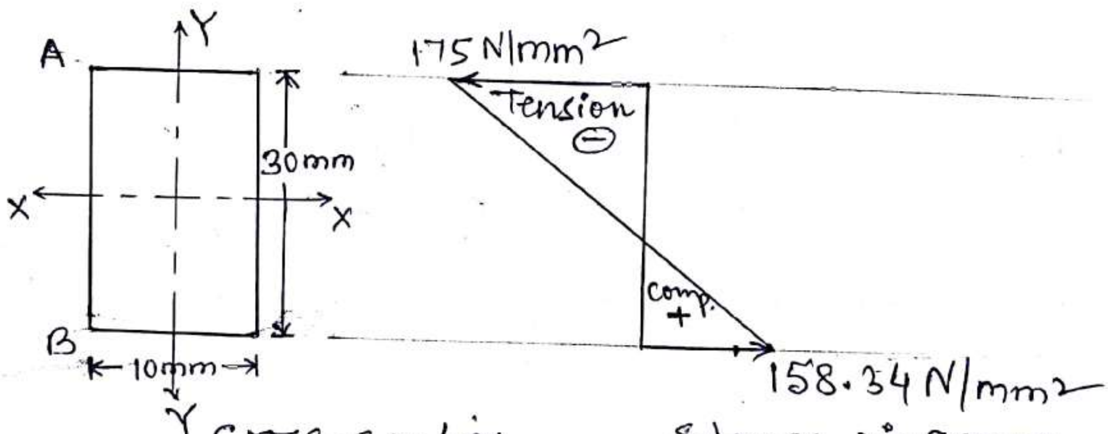
$$\sigma_{max} = \sigma_0+\sigma_b = 2.73+6.98 = 9.71\ N/mm^2\ \text{(compressive)}$$

$$\sigma_{min} = \sigma_0-\sigma_b = 2.73-6.98 = \underline{-4.25\ N/mm^2}\ \text{(Tensile)}$$

Q9. A C-Clamp made up of rectangular cross section 30 × 10 mm as shown in figure is subjected to a force of 2.5 kN. Find the stresses induced at section AB.

Solution :–





Cross-section   Stress Diagram

Given: Width, $b = 10$ mm

depth, $d = 30$ mm

Load, $P = 2.5\,kN = 2.5 \times 10^3\,N$

Eccentricity, $e = 100$ mm

Find: σ_{max} and σ_{min}.

We know that

$$\sigma_{max} = \sigma_0 + \sigma_b$$

$\sigma_{min} = \sigma_0 - \sigma_b$

Direct stress, $\sigma_0 = \frac{P}{A}$

cross-sectional area, $A = b \times d$

$= 10 \times 30$

$\boxed{A = 300\ mm^2}$

$\sigma_0 = \frac{P}{A} = \frac{2.5 \times 10^3}{300}$

$\boxed{\sigma_0 = 8.33\ N/mm^2\ (\text{Tensile})}$

The load is eccentric with respect to xx axis.

Bending stress, $\sigma_b = \frac{M}{Z_{xx}}$

$M = P \times e = 2.5 \times 10^3 \times 100$

$\boxed{M = 2.5 \times 10^5\ N\text{-}mm}$

$Z_{xx} = \frac{bd^2}{6} = \frac{10 \times 30^2}{6}$

$Z_{xx} = 1500\ mm^3$

$\sigma_b = \frac{M}{Z_{xx}} = \frac{2.5 \times 10^5}{1500}$

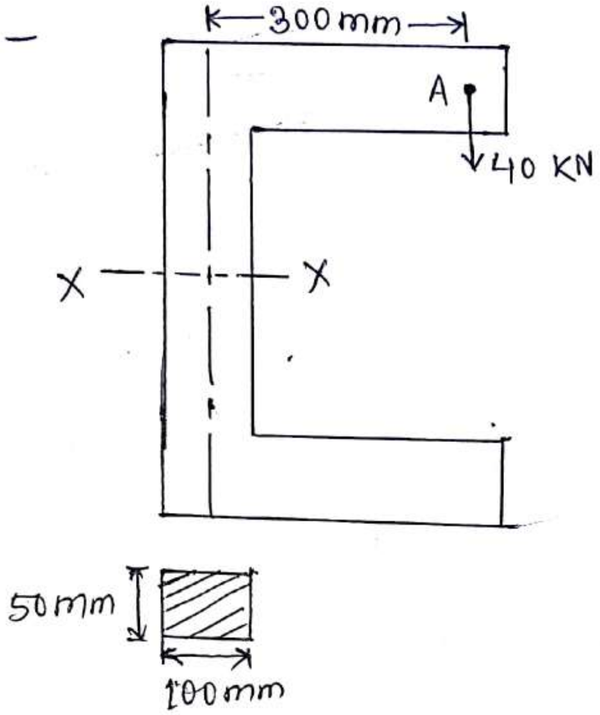
$\boxed{\sigma_b = 166.67\ N/mm^2}$

$\sigma_{max} = \sigma_0 + \sigma_b = 8.33 + 166.67 = 175\ N/mm^2$ (Tensile) {on face A}

$\sigma_{min} = \sigma_0 - \sigma_b = 8.33 - 166.67 = -158.34\ N/mm^2$ (Compressive) on face B.

(Q.10.) A rectangular rod of size 50 mm×100 mm is bent into 'C' shape as shown in figure and applied load of 40 KN at point 'A'. calculate the resultant stresses developed at section XX.

Solution:-



Given:

Width, $b = 100$ mm

depth, $d = 50$ mm

Load, $P = 40$ KN $= 40\times10^3$ N

eccentricity, $e = 300$ mm

Find:- σ_{max} and σ_{min}

We Know that

$$\sigma_{max} = \sigma_0 + \sigma_b$$

$$\sigma_{min} = \sigma_0 - \sigma_b$$

Direct Stress, $\sigma_0 = \frac{P}{A}$

Cross-sectional Area, $A = b \times d$

$= 100 \times 50$

$\boxed{A = 5000 \text{ mm}^2}$

$\sigma_0 = \frac{P}{A} = \frac{40 \times 10^3}{5000} = \underline{8 \text{ N/mm}^2}$ (Tensile)

The load is eccentric with respect to YY-axis.

Bending Stress, $\sigma_b = \frac{M}{Z_{YY}}$

$M = P \times e = 40 \times 10^3 \times 300$

$M = 12 \times 10^6$ N-mm

Section Modulus, $Z_{YY} = \frac{db^2}{6} = \frac{50 \times (100)^2}{6}$

$Z_{YY} = \frac{5}{6} \times 10^5 \text{ mm}^2$

$\sigma_b = \frac{M}{Z_{YY}}$

$= \frac{12 \times 10^6}{\frac{5}{6} \times 10^5}$

$\boxed{\sigma_b = 144 \text{ N/mm}^2}$

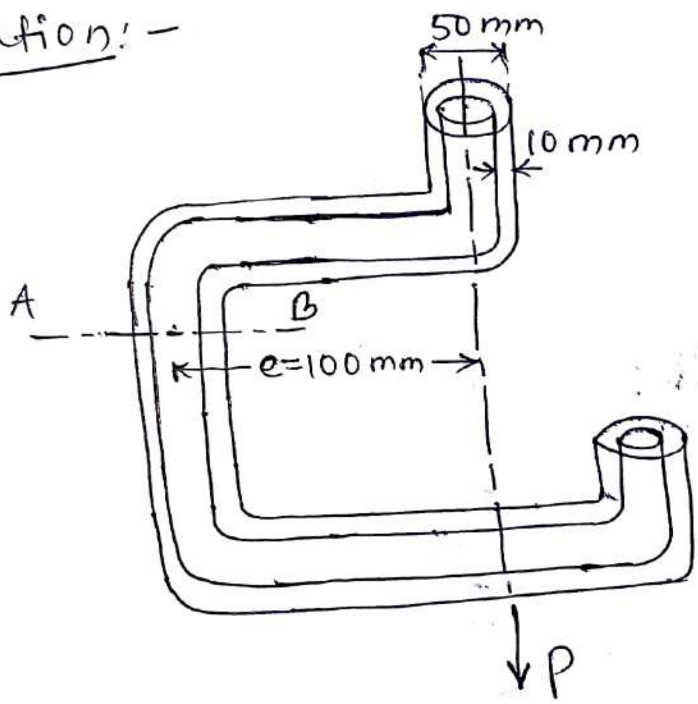
Resultant Stress:-

$\sigma_{max} = \sigma_0 + \sigma_b = 8 + 144 = 152 \text{ N/mm}^2$ (Tensile)

$\sigma_{min} = \sigma_0 - \sigma_b = 8 - 144 = -136 \text{ N/mm}^2$ (compressive)

Q11. A mild steel tube of 50 mm external diameter and 10 mm thickness is bent in the form of hook as shown in figure. What maximum load 'P' the hook can lift if the stresses on the cross-section AB should not exceed 100 MPa in tension and 25 N/mm² in compression?

Solution:-



Given:-

External diameter, $D = 50$ mm

Internal diameter, $d = D - 2t$ (t = Thickness)

$= 50 - 2 \times 10$

$d = 30$ mm

$\sigma_{max} = 100$ MPa $= 100$ N/mm² (Tensile)

$\sigma_{min} = -25$ N/mm² (compressive)

Find:- Maximum load, $P = ?$

we know that,

$$\sigma_{max} = \sigma_0 + \sigma_b$$

$$\sigma_{max} = \frac{P}{A} + \frac{M}{Z}$$

$$\sigma_{max} = \frac{P}{\frac{\pi}{4}(D^2-d^2)} + \frac{P \times e}{\frac{\pi}{32}\left(\frac{D^4-d^4}{D}\right)}$$

$$\therefore 100 = \frac{P}{\frac{\pi}{4}(50^2-30^2)} + \frac{P \times 100}{\frac{\pi}{32}\left(\frac{50^4-30^4}{50}\right)}$$

$$\therefore 100 = 0.010156\,P$$

$$\therefore \boxed{P = 9.85\ KN} \text{ —— (1)}$$

$$\sigma_{min} = \sigma_0 - \sigma_b$$

$$\sigma_{min} = \frac{P}{A} - \frac{M}{Z}$$

$$\therefore -25 = \frac{P}{\frac{\pi}{4}(50^2-30^2)} - \frac{P \times 100}{\frac{\pi}{32}\left(\frac{50^4-30^4}{50}\right)}$$

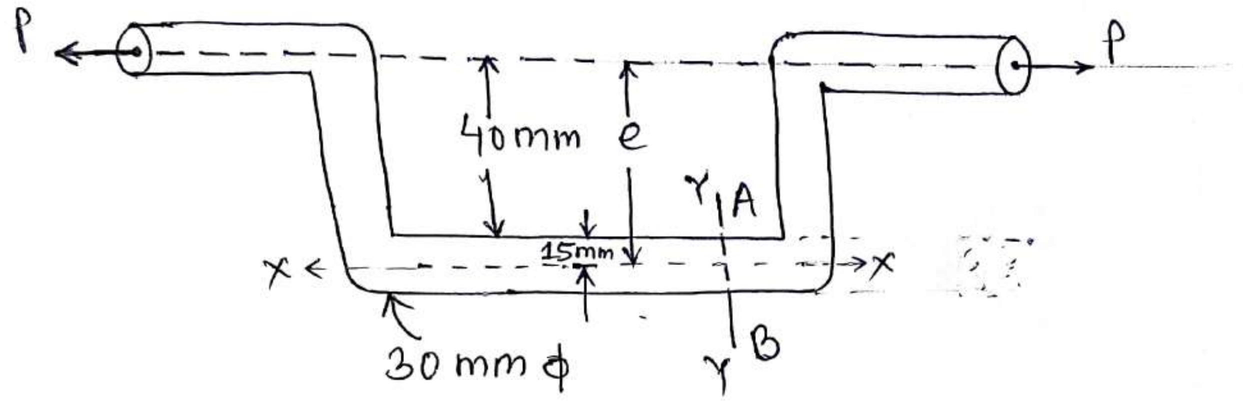
$$\therefore -25 = -0.000564\,P$$

$$\therefore \boxed{P = 2919.2N = 2.92KN} \text{ —— (2)}$$

From equation (1) and (2), the safe load to satisfy the required stress criteria is the lesser value of (1) & (2),

$\boxed{P = 2.92KN}$ Ans

Q12. A 30mm diameter rod is bent up to form an offset link as shown in figure. If permissible tensile stress is 80 MPa, determine the maximum value of P.



Solution: –

Given: diameter of rod, d = 30 mm

Eccentricity, $e = 40 + \frac{d}{2}$

$= 40 + \frac{30}{2}$

$e = 55\,mm$

$\therefore \sigma_{max} = 80\,MPa\ (tensile) = 80\,N/mm^2\ (tensile)$

Find: Load, P.

We know that

$$\sigma_{max} = \sigma_0 + \sigma_b$$

$$= \frac{P}{A} + \frac{M}{Z_{xx}}$$ [The load is eccentric w.r.t. xx-axis)

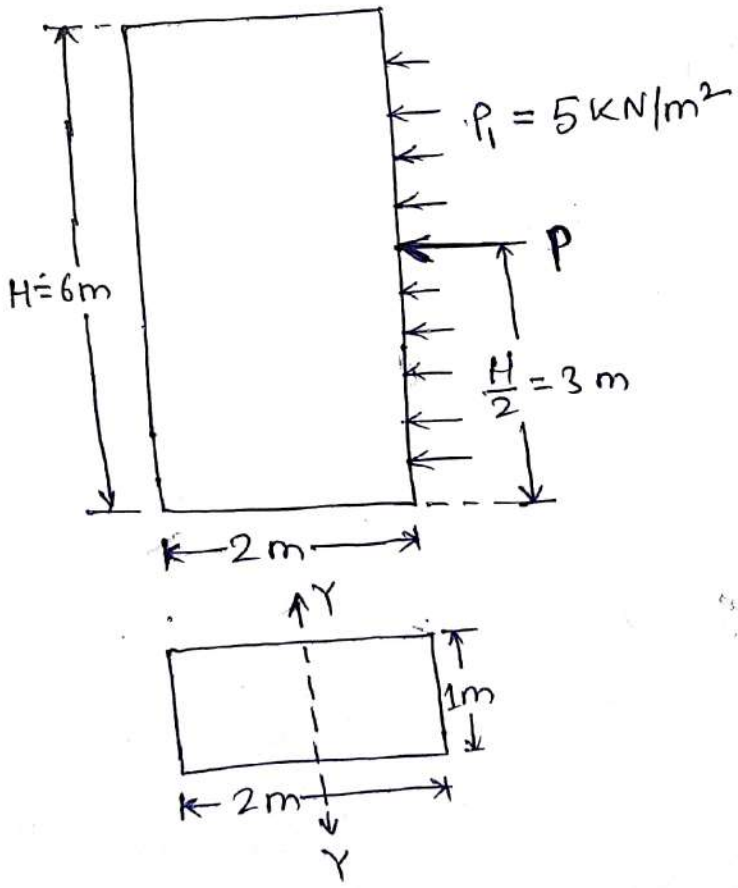
$$\therefore \sigma_{max} = \frac{P}{\frac{\pi}{4}d^2} + \frac{P \times e}{\frac{\pi}{32}d^3}$$

$$\therefore 80 = \frac{P}{\frac{\pi}{4} \times 30^2} + \frac{P \times 55}{\frac{\pi}{32} \times 30^3}$$

$$\therefore 80 = 0.022164\,P$$

$$\therefore \boxed{P = 3609.46\ N = 3.6\,KN}$$ Ans

Q13. A masonry wall 6 m high, 2 m thick and 1m wide is subjected to a horizontal wind pressure of 5 KN/m2 on 1 m face. Find the value of net stresses at base of the wall. Density of masonry is 20 KN/m3.



Solution:-

Given:- Height of masonry wall, $H = 6m$

Thickness, $t = 2m$

Width, $b = 1m$

Horizontal wind pressure, $P_1 = 5 KN/m^2 = 5\times10^3 N/m^2$

Density of masonry, $\rho = 20 KN/m^3 = 20\times10^3 N/m^3$

Find:- σ_{max} and σ_{min}.

(i) Weight of wall, $W = A\times H\times\rho$ (A = cross-sectional Area of wall.)

$= (t\times b)\times H\times\rho$

$= 2\times1\times6\times20\times10^3$

$$\boxed{W = 24\times10^4 N}$$

(ii) cross-sectional Area, $A = t\times b$

$= 2\times1$

$$\boxed{A = 2m^2}$$

(iii) Direct stress, $\sigma_0 = \frac{W}{A} = \frac{24\times10^4}{2}$

$$\boxed{\sigma_0 = 12\times10^4 N/m^2}$$

(iv) Total wind load,

P = Wind pressure x projected Area

$= P_1\times A_1$

$= P_1\times(b\times H)$

$P = 5\times10^3 \times (1\times6)$

$\boxed{P = 30\times10^3\ N}$

(v) Moment of P about the base,

$M = P \times \frac{H}{2}$

$= 30\times10^3 \times \frac{6}{2}$

$\boxed{M = 90\times10^3\ N\text{-}m}$

(vi) The load is eccentric about YY-axis.

Bending Stress, $\sigma_b = \frac{M}{Z_{YY}} = \frac{M}{\frac{bt^2}{6}} = \frac{90\times10^3}{\frac{1\times2^2}{6}}$

$\boxed{\sigma_b = 134328.36\ N/m^2}$

$\sigma_{max} = \sigma_0 + \sigma_b = 12\times10^4 + 134328.36 = 254328.36\ N/m^2$ (Compressive)

Ans

$\sigma_{min} = \sigma_0 - \sigma_b = 12\times10^4 - 134328.36 = -14328.36\ N/m^2$ (Tensile)

Bending Moment and Shear Force

Beam:- It is a structural member which is acted upon by a system of external loads at right angles to the axis.

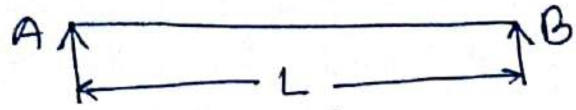
Point load:- A point load is one which is Considered to act at a point.
It is also called as concentrated load.

Distributed load:- A distributed load is one which is distributed or spread in some manner over the length of the beam.
If the spred is uniform it is said to be uniformly distributed load and is abbreviated as U.D.L.
If the spread is not at uniform rate, it is said to be non-uniformly distributed load. Triangular and trapezium distributed loads fall under this category.

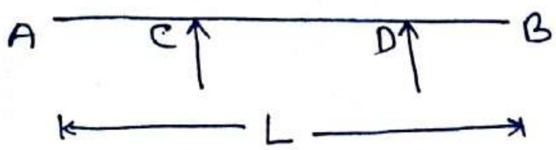
Classification of beams

(i) **Cantilever beam:-** A Cantilever beam is a beam whose one end is fixed and the other end free.

2. Simply (or freely) Supported beam :- A beam which ends freely rest on walls or Columns or knife edges is called as simply supported beam.

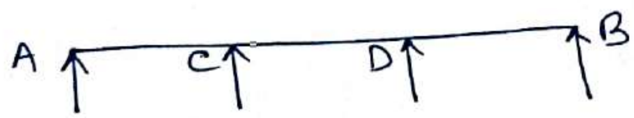


3. Overhanging beam :- If the end portion of the beam extends beyond the support, it is called as an overhanging beam.



4. Fixed beam :- A beam whose both ends are rigidly fixed in walls or columns, it is called as fixed beam.

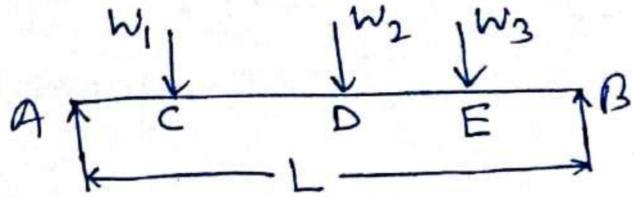
5. Continuous beam :- A beam which has more than two supports, it is called as Continuous beam.



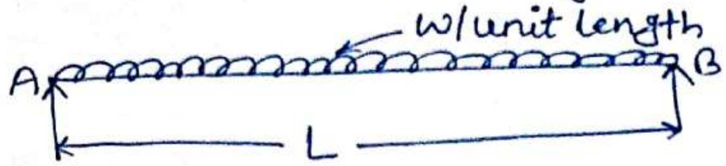
Here A and B are end supports, where as C and D are intermediate supports.

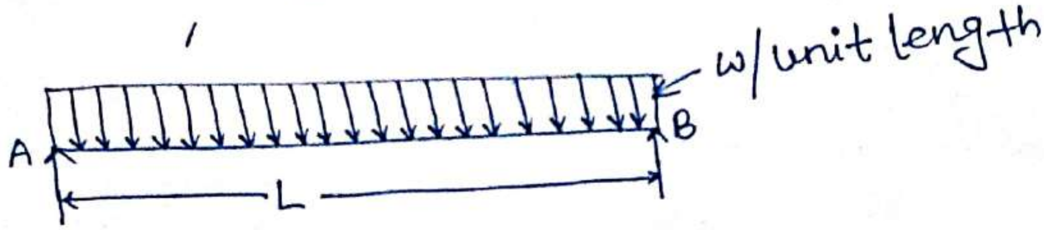
Types of loads

(1) **Point Load**:- A load acting at a point on the beam is known as point load. It is also called as concentrated load.



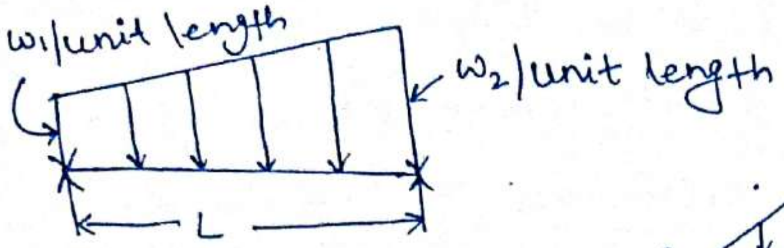
(2) **Uniformly distributed load (u.d.l.)**:- A load which is spread up uniformly on the beam is known as uniformly distributed load.



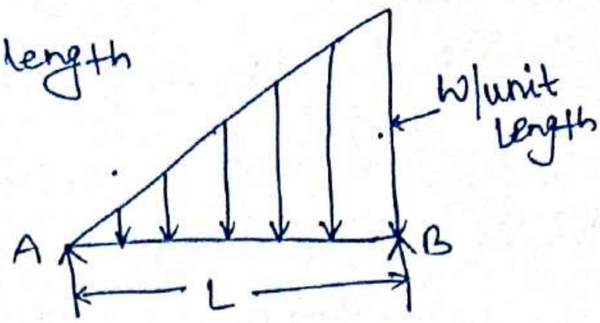


Here w/unit length is called as intensity of u.d.l.

(3) **Uniformly varying Load**:- A load which is spread up in a non-uniform manner i.e. intensity of load changes continuously but the rate of change is uniform on each unit length, then it is called a uniformly varying load and is written as u.v.l.



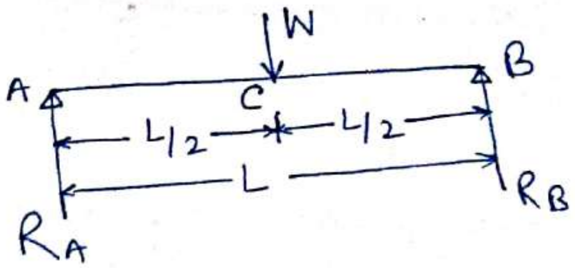
Trapezoidal load



Triangular load

Reactions of Simply Supported Beams

Case I :- A simply supported beam of span 'L' carrying point load 'W' as shown in fig.



As there is no horizontal force acting on the beam.

$$\therefore \Sigma F_x = 0$$

NOW $\Sigma F_Y = 0$ [Assuming upward forces positive and downward forces negative]

$$\therefore R_A + R_B - W = 0$$

$$\therefore R_A + R_B = W \quad \text{——— (i)}$$

Moment about A is zero i.e. $\Sigma M = 0$

$R_A \times 0 + W \times \frac{L}{2} - R_B \times L = 0$ [Taking clockwise moment positive and anticlockwise moment negative]

$$\therefore \frac{WL}{2} = R_B L$$

$$\therefore \frac{W}{2} = R_B$$

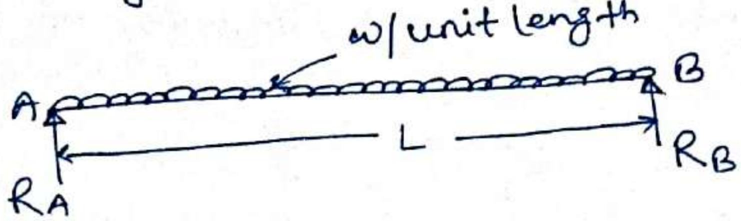
$$\therefore \boxed{R_B = \frac{W}{2}} \quad \text{——— (ii)}$$

Substituting the value of R_B in eqn (i), we get

$$R_A + \frac{W}{2} = W$$

$$\boxed{R_A = \frac{W}{2}}$$

Case-II: A simply supported beam of span 'L' carrying u.d.l. w/unit length over the entire span as shown in fig.



We have to find beam reactions R_A and R_B.

As there is no horizontal force acting on the beam. $\therefore \Sigma F_x = 0$

At equilibrium,

$$\Sigma F_Y = 0$$

$$\therefore R_A + R_B - w \times L = 0$$

$$\therefore R_A + R_B = wL \qquad \text{——— (i)}$$

Moment about point A is zero. i.e. $\Sigma M_A = 0$

$$\therefore R_A \times 0 + wL \times \frac{L}{2} - R_B \times L = 0$$

[Taking clockwise moment positive & anticlockwise moment negative]

$$\therefore \frac{wL^{\cancel{2}}}{2} = R_B \cancel{L}$$

$$\therefore \frac{wL}{2} = R_B$$

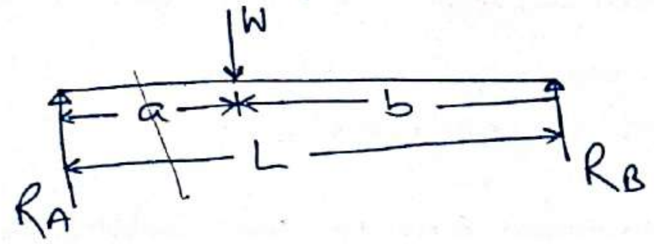
$$\therefore \boxed{R_B = \frac{wL}{2}} \qquad \text{——— (ii)}$$

Substituting the value of R_B in eqn (i), we get.

$$R_A + \frac{wL}{2} = wL$$

$$\therefore \boxed{R_A = \frac{wL}{2}}$$

Case III: A simply supported beam of span 'L' carrying an eccentric point load 'W' as shown in fig.



We have to find beam reactions R_A & R_B.

At equilibrium, $\Sigma F_x = 0$ and $\Sigma F_y = 0$

As there is no horizontal force acting on the beam.

Now, $\Sigma F_y = 0$

$$\therefore R_A + R_B - W = 0 \quad [\uparrow +, \downarrow -]$$

$$\therefore R_A + R_B = W \quad \text{——— (i)}$$

Applying $\Sigma M_A = 0$ $(\circlearrowright +, \circlearrowleft -)$

$$\therefore R_A \times 0 + W \times a - R_B \times L = 0$$

$$\therefore Wa = R_B L$$

$$\therefore \boxed{R_B = \frac{Wa}{L}} \quad \text{——— (ii)}$$

Substituting the value of R_B in eqn (i), we get-

$$R_A + \frac{Wa}{L} = W$$

$$\therefore R_A = W - \frac{Wa}{L}$$

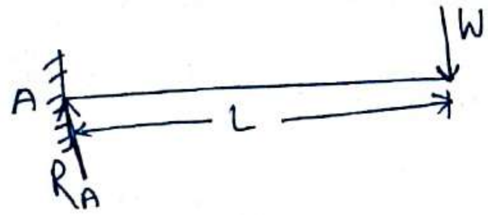
$$\therefore R_A = W\left[1 - \frac{a}{L}\right]$$

$$\therefore R_A = W\left[\frac{L-a}{L}\right]$$

$$\boxed{R_A = \frac{wb}{L}} \quad \left(\begin{array}{l} a+b = L \\ b = L-a \end{array}\right)$$

Reactions of Cantilever beam

Case I :- A Cantilever beam of span L carrying a point load 'W' at its free end as shown in fig.

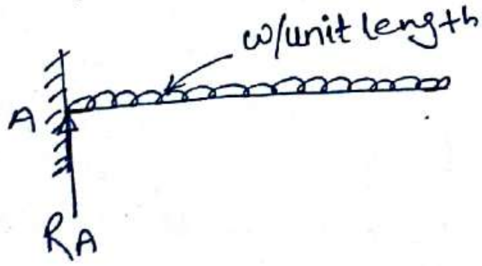


$$\Sigma F_y = 0$$

$$R_A - W = 0$$

$$\boxed{R_A = W}$$

Case II: A Cantilever beam of span 'L' carrying a u.d.l. w/unit length over the entire span as shown in fig.

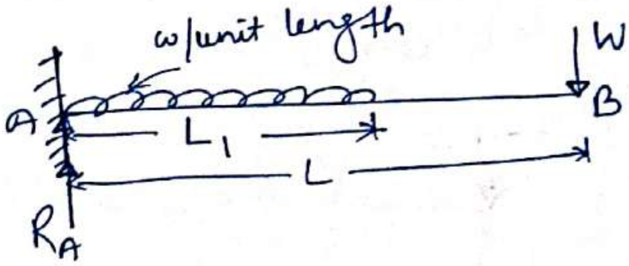


$$\Sigma F_y = 0$$

$$\therefore R_A - wL = 0$$

$$\therefore \boxed{R_A = wL}$$

Case III. A Cantilever beam of span 'L' carrying a point load 'W' at its free end and also carrying a u.d.l. 'w'/unit length over the span in some portion as shown in fig.



We have to find the reaction R_A.

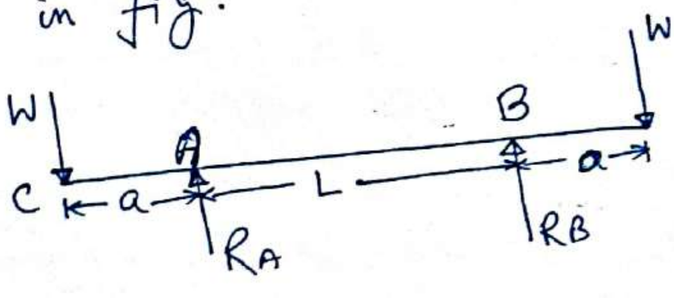
$$\Sigma F_Y = 0$$

$$\therefore R_A - wL_1 - W = 0$$

$$\therefore \boxed{R_A = wL_1 + W}$$

✱ Reactions of overhanging beams

Case I: A overhanging beam of span 'L' carrying point loads over the span as shown in fig.



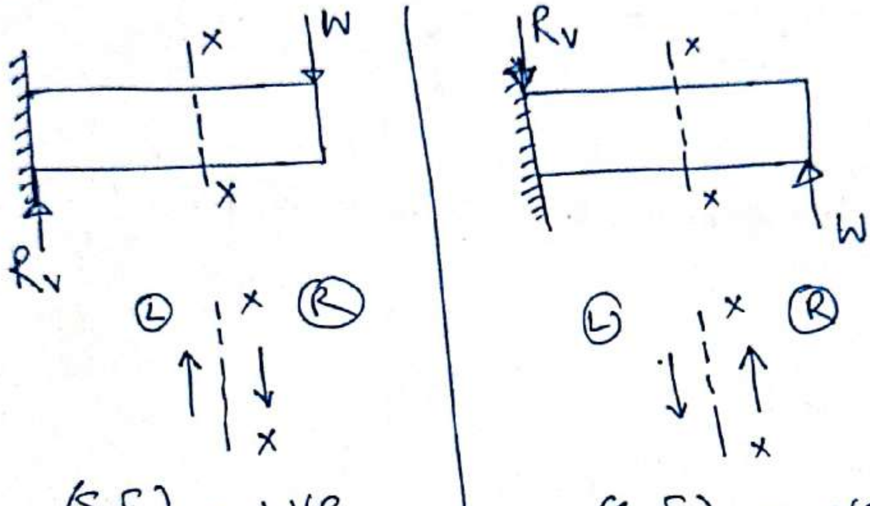
$$\Sigma F_Y = 0$$

$$\therefore R_A + R_B = W + W = 2W$$

$\boxed{R_A = R_B = W}$ (As the loading is symmetrical)

Shear force :- Shear force at any cross-section of the beam is the algebraic sum of all vertical forces on the beam acting on the right or left side of the section.

Sign convention for shear force :-



$(S.F.)_{xx} = +ve$ $(S.F.)_{xx} = -ve$

L = Left side of section xx.

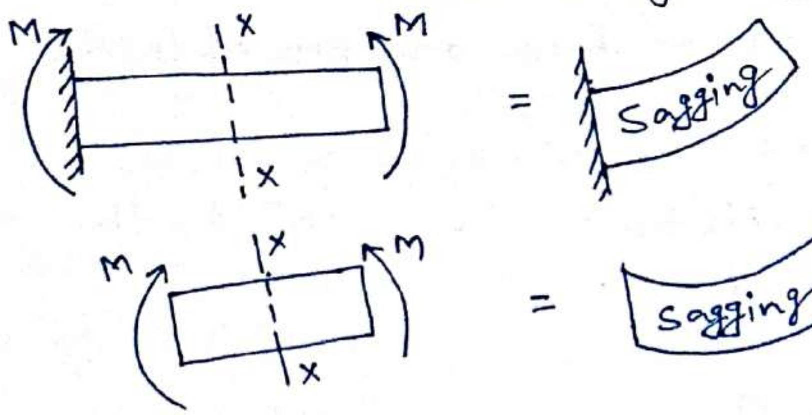
R = Right side of section xx

- An upward force to the left of the section and downward force to the right of the section is taken as positive and its vice-versa is taken as negative.

Bending Moment:- Bending moment at any cross-section of the beam is the algebraic sum of the moments of all the forces acting on the right or left side of the section.

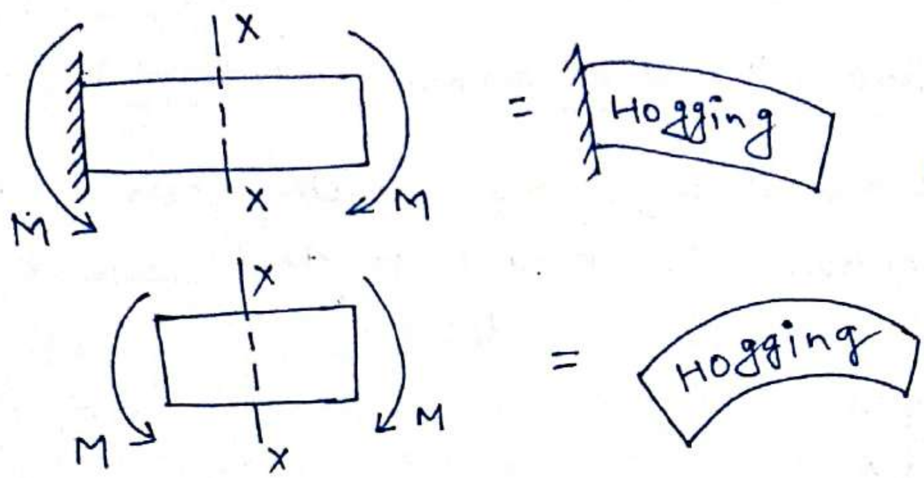
Sign Convention for Bending Moment:-

(i)



Bending Moment = positive

(ii)



Bending Moment = Negative

Note:-

(i) If a clockwise bending moment is acting on the left hand side of the section and anticlockwise bending moment on the right hand side of the section, the bending moment on that section is said to be positive.

(ii) If an anticlockwise bending moment

is acting on the left hand side of the section and a clockwise bending moment is acting on the right hand side section, then the bending moment of that section is said to be negative.

Relation between shear force and rate of loading

The rate of change of shear force with respect to the distance is equal to the intensity of loading.

$$\frac{dF}{dx} = \omega$$

Relation between bending Moment and shear force.

The rate of change of bending moment at any section is equal to the shear force at that section.

$$\frac{dM}{dx} = F$$

Note:- If $F = 0$, $\frac{dM}{dx} = 0$, it means the bending moment will be maximum.

The point, at which shear force is zero or shear force changes the sign from positive to negative, is a point of maximum bending moment.

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The Point of Contraflexure

The bending moments of opposite nature always produce curvatures of beam in opposite directions. In a beam if the bending moment changes sign at a point, the point itself having zero bending moment, the beam changes curvature at this point of zero bending moment and this point is called the point of contraflexure.

The point of contraflexure is also called the point of inflexion or a virtual hinge.

Important points to draw shear force diagram (S.F.D.) and bending Moment diagram (B.M.D.)

The following points should be kept in mind while drawing S.F.D. and B.M.D.

(i) Base of S.F.D. and B.M.D is equal to the span of the beam.

(ii) positive values of S.F. and B.M. are plotted above the base line and negative values below the base line.

(iii) The values of S.F. and B.M. must be calculated at all critical points and written near the respective ordinates.

Such critical points are a point where u.d.l. starts and ends, a point where the concentrated load acts on the beam, a point where S.F. changes its sign (means S.F. is zero.)

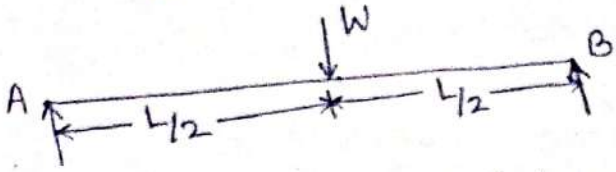
(iv) In case of overhanging beam, a point of Contraflexure (i.e. a point of zero B.M.) must be located. This point can be located by equating the expression of bending moment to zero and solving it.

(v) The location of point of zero S.F. and the point of contraflexure must be marked from the supports or from the ends of the beam whichever is convenient.

(vi) S.F.D. is drawn below the loaded beam and B.M.D below the S.F.D.

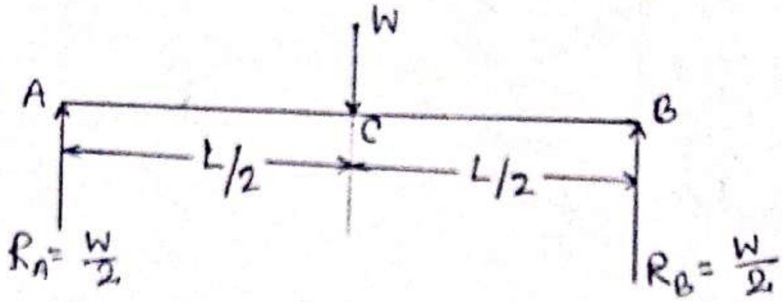
S.F.D. and B.M.D. For Simply Supported Beams

Case 1: A Simply supported beam of span 'L' carrying a central load 'W'.

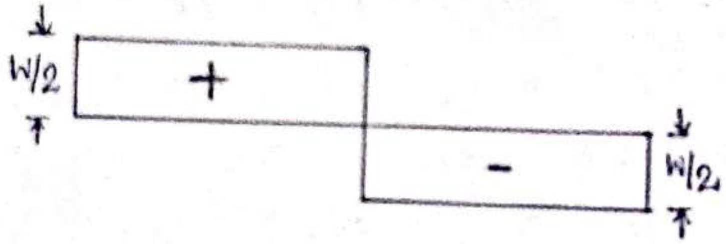


(i) Simply supported beam

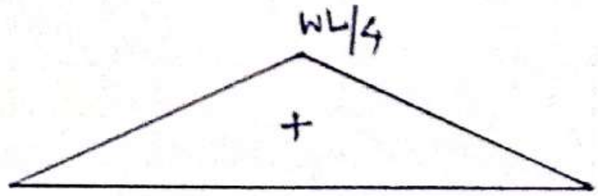
Solution :-



(i) simply supported beam



(ii) S.F.D.



(iii) B.M.D.

Since the load is at the ~~support~~ Centre, the reactions at the supports are equal.

$$\therefore R_A = R_B = W/2$$

Shear Force (S.F.) calculations:-

S.F. at any section between A and C is

$$F_x = +R_A = \frac{W}{2}$$

S.F. at any section between C and B is

$$F_x = -R_B = -\frac{W}{2}$$

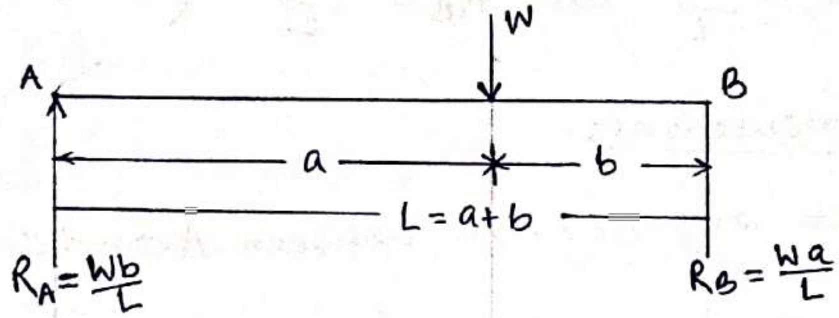
B.M. calculations:-

$M_A = M_B = 0$ at simply supported ends.

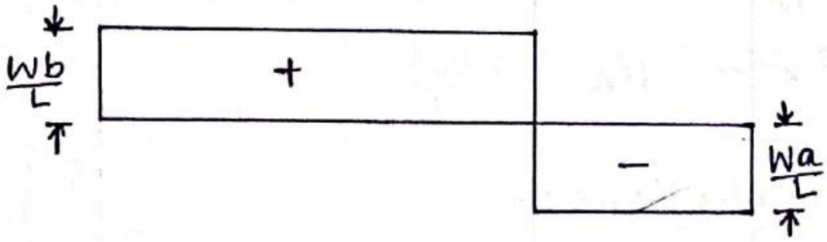
Since S.F. changes its sign from positive to negative at point C, the maximum B.M. will occur at C.

$$\therefore M_{max} = M_C = +\frac{W}{2} \times \frac{L}{2} = \frac{WL}{4}$$

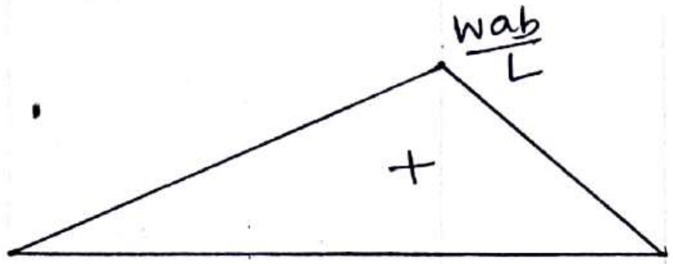
Case 2: A simply supported beam of span 'L' Carrying an eccentric point load:



(i) Simply supported beam



(ii) S.F.D.



(iii) B.M.D.

Supports reactions are:-

$$R_A = \frac{Wb}{L} \quad \text{and} \quad R_B = \frac{Wa}{L}$$ (As already seen)

S.F. Calculations:-

S.F. at any section between A and C is,

$$F_x = +R_A = \frac{Wb}{L}$$

S.F. at any section between C and B is;

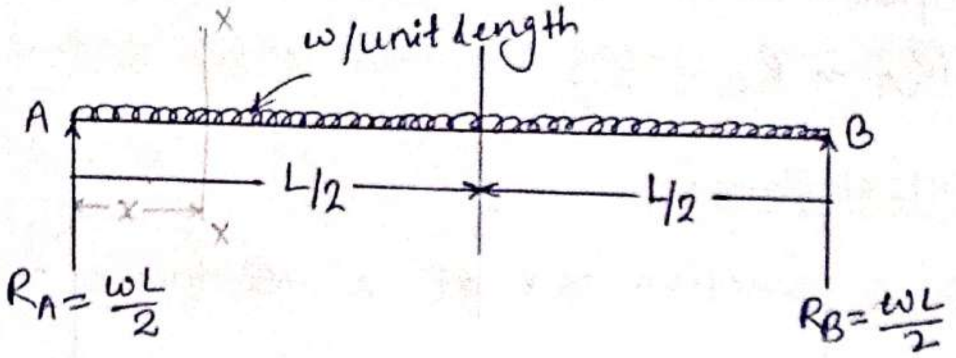
$$F_x = -R_B = -\frac{Wb}{L}$$

B.M. Calculations:-

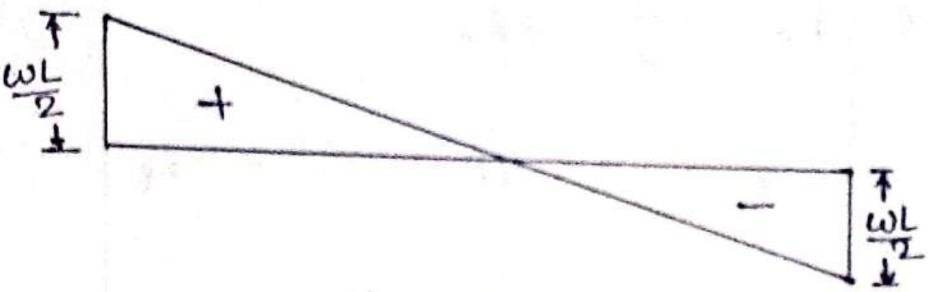
$M_A = M_B = 0$, at simply supported ends.

$$M_C = M_{max} = +\frac{Wb}{L} \times a = \frac{Wab}{L}$$

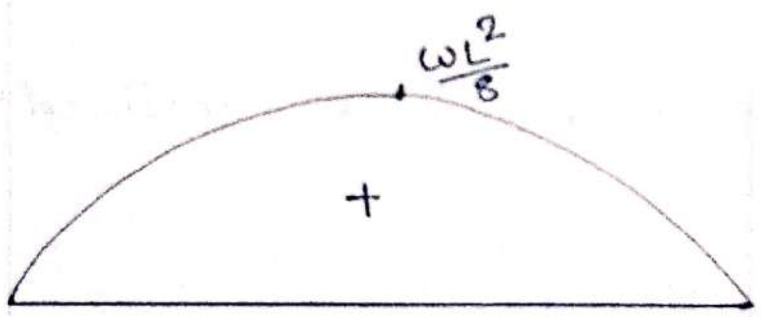
Case 3: A simply supported beam of span 'L' carrying a u.d.l. w/unit length over the entire span.



(i) Simply supported beam



(ii) S.F.D.



(iii) B.M.D.

Support reactions:

Since the load is uniformly distributed over the entire length of the beam, the reactions are equal.

$$R_A = R_B = \frac{wL}{2}$$

S.F. Calculations:

Take a section XX at a distance x from A.

$$F_x = \text{S.F. at Section XX} = \frac{wL}{2} - wx$$

At $A, x=0$, $F_A = \frac{wL}{2} = +R_A$

At B, $x = L$, $F_B = \frac{wL}{2} - wL = -\frac{wL}{2} = -R_B$

At the centre of the beam. i.e. at C, $x = L/2$

$$F_C = 0$$

In this case, S.F.D. is an inclined straight line.

B.M. calculations:

B.M. at a section XX at a distance x from A is given by,-

$$M_x = \frac{wL}{2} \times x - wx \times \frac{x}{2} = \frac{wL}{2}x - \frac{wx^2}{2}$$

Since the power of x is 2, the variation of B.M. is parabolic.

At A, $x = 0$, $M_A = 0$

At B, $x = L$, $M_B = 0$

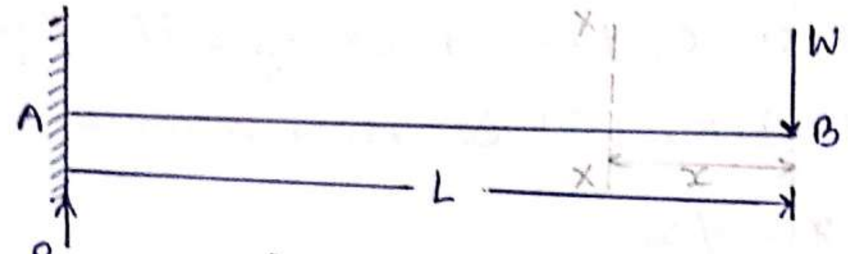
At the centre C, the S.F. changes its sign and hence the B.M. will be maximum at 'C'.

At C, $x = L/2$

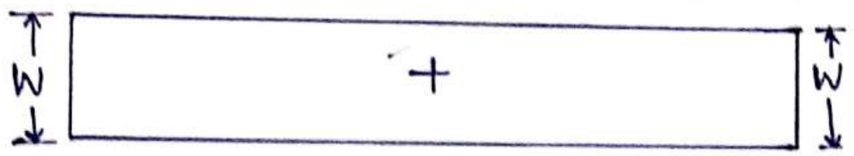
$$M_C = M_{max} = \frac{wL^2}{8}$$

S.F.D. and B.M.D. for Cantilever beams

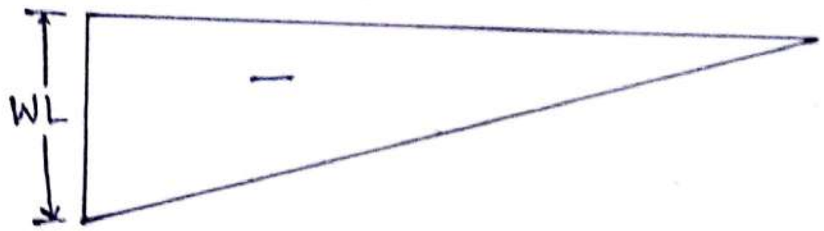
Case 1: A Cantilever beam carrying a point load at its free end



(i) Cantilever beam



(ii) S.F.D.



(iii) B.M.D.

Let us consider a cantilever beam AB of span 'L' carrying a point load W at its free end B.

Take a section XX at a distance x from the free end B.

F_x = S.F. at section XX = +W

Thus S.F. is constant for all sections between A and B.

NOW, B.M. at section XX $= M_x = -W \times x$

Since x is measured from B, B is taken as origin.

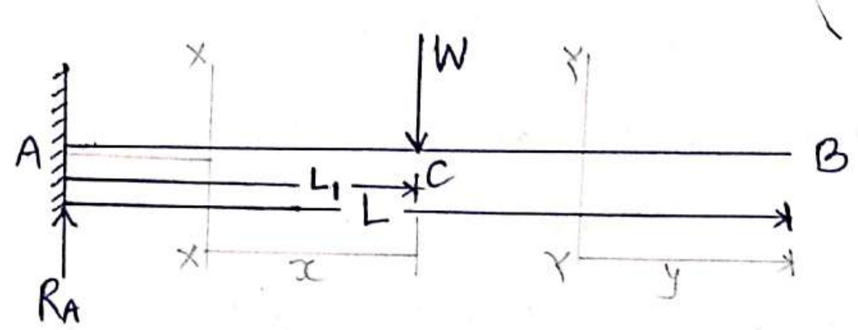
At B, $x = 0$

$\therefore M_B = -W \times 0 = 0$

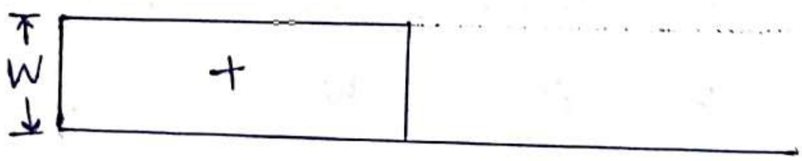
At A, $x = L$

$\therefore M_A = -WL$

Case 2: A cantilever beam carrying a point load not at the free end.



(i) cantilever beam



(ii) S.F.D.

(iii) B.M.D.

Let us consider a cantilever beam AB of span 'L' Carrying a point Load 'W' at a point C at a distance L_1 from the fixed end.

Take a section YY at a distance y from the free end B in portion CB.

F_y = S.F. at section YY = 0 [Since no force to the right of section YY)

∴ Zero S.F. for all the sections between C and B.

Take a section XX at a distance x from the point C in portion AC.

F_x = S.F. at section XX = +W

∴ S.F. for all sections between A and C is Constant and it is +W.

B.M. Calculations:-

M_y = B.M. at section YY = 0 [Since there is no force to the right of section YY]

∴ B.M. is Zero for all the sections between C and B.

NOW, M_x = B.M. at section XX at a distance x from C.
$= -W \times x$

Since x is measured from C, C is taken as origin.

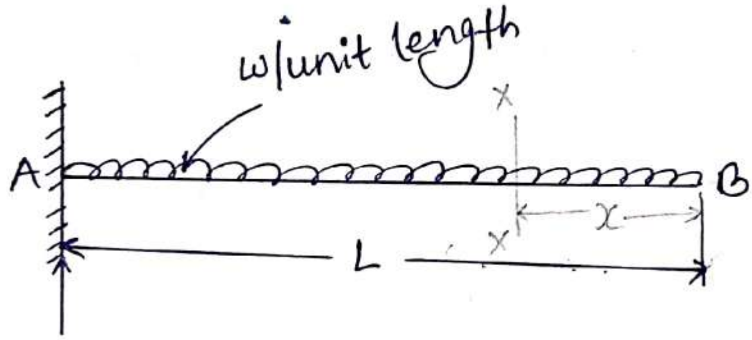
At C, $x = 0$

$\therefore M_C = 0$

At A, $x = L_1$

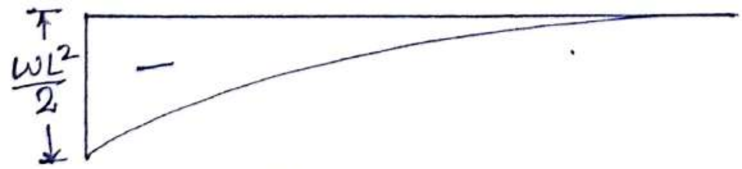
$\therefore M_A = -WL_1$

Case 3: A cantilever beam carrying a u.d.l. w/unit length over the entire span:



(i) Cantilever beam

(ii) S.F.D.



(iii) B.M.D.

Let us consider a cantilever beam AB of span 'L' carrying a u.d.l. w/unit length over the entire span AB.

Take a section XX at a distance x from the free end B.

F_x = S.F. at section XX = $+w \times x$

Since x is measured from B, B is taken as origin.

At B, $x = 0$

$\therefore F_B = 0$

At A, $x = L$

$\therefore F_A = +w \times L$

B.M. Calculations:

M_x = B.M. at section XX $= -wx \times \frac{x}{2}$

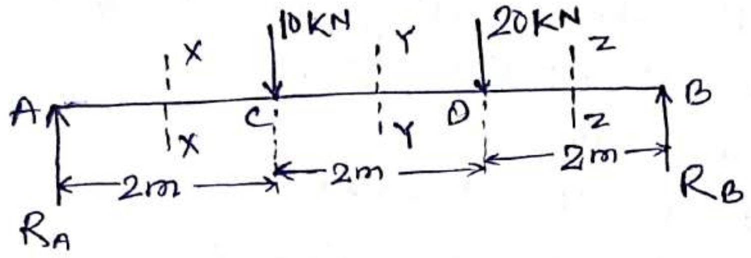
At B, $x = 0$,

$\therefore M_B = 0$

At A, $x = L$

$\therefore M_A = -\frac{wL^2}{2}$

Concept of Shear force and Bending Moment



Consider a simply supported beam of span 6m carrying two point loads 10KN and 20KN at 2m and 4m from left support respectively.

Support reactions:- $\Sigma F_Y = 0$

$$R_A - 10 - 20 + R_B = 0$$

$$R_A + R_B = 30 \quad \text{——— (i)}$$

And, $\Sigma M_A = 0$

$$10 \times 2 + 20 \times 4 - R_B \times 6 = 0$$

$$\therefore \quad 20 + 80 - 6R_B = 0$$

$$\therefore \quad 100 - 6R_B = 0$$

$$\therefore \quad R_B = \frac{100}{6} = \frac{50}{3} \text{ KN}$$

$$\therefore \boxed{R_B = \frac{50}{3} = 16.67 \text{KN}}$$

From (i), $\boxed{R_A = 30 - 16.67 = 13.33 \text{ KN}}$

Consider a section XX in portion AC,

By considering the forces acting on the left side of section XX, S.F. at section XX,

$$F_X = R_A = 13.33 \text{ KN}$$

By considering the forces acting on the right side of section XX, S.F. at section XX,

$$F_X = 10 + 20 - R_B$$
$$= 10 + 20 - 16.67$$
$$= 13.33 \text{ KN}$$

Consider a section YY in portion CD,
By considering the forces acting on the left side of section YY, S.F. at section YY,

$$F_Y = R_A - 10$$
$$= 13.33 - 10 = 3.33 \text{ KN}$$

By considering the forces acting on the right side of Section YY, S.F. at section YY,

$$F_Y = 20 - R_B = 20 - 16.67 = 3.33 \text{ KN}$$

Now consider a section ZZ in portion DB,
By considering the forces acting on Left side of section ZZ, S.F. at section ZZ, S.F. at section ZZ,

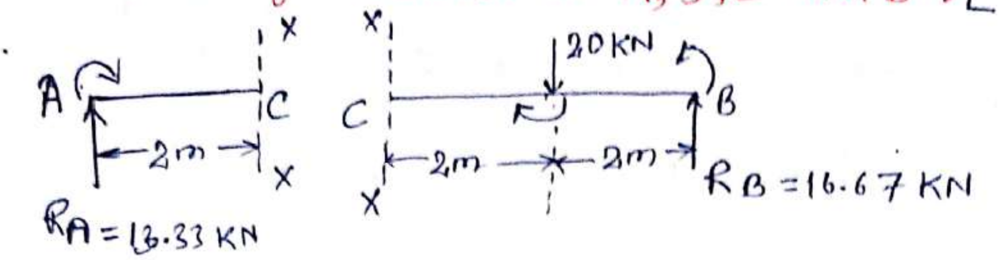
$$F_Z = R_A - 10 - 20 = 13.33 - 30 = 16.67 \text{ KN}$$

By considering the forces acting on right side of section ZZ, S.F. at section ZZ, S.F. at section ~~YY~~ ZZ,

$$F_Z = -R_B = -16.67 \text{ KN}.$$

Note:- It is very clear that the shear force at any cross-section of the beam is the algebraic sum of all vertical forces acting on right or left side of the section.

To find bending moments at A, B, C and D : [$\overset{+}{\smile}$, $\overset{-}{\frown}$]



To find M_C:

Take a section XX at C and draw the free body diagram at C as shown in above fig.

By considering the moments of all forces acting on the left side of C,

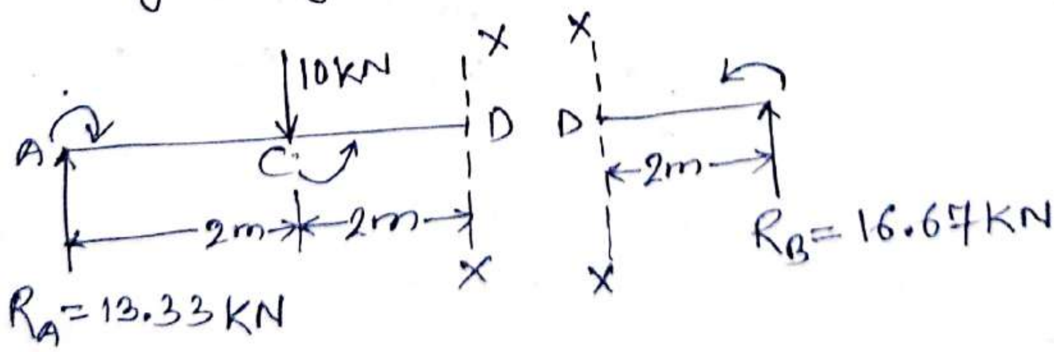
$$M_C = 13.33 \times 2 = 26.66 \text{ KN-m}$$

By Considering the moments of all forces acting on the right side of C,

$$M_C = -20 \times 2 + 16.67 \times 2 = 26.68 \text{ KN-m}.$$

To Find M_D:

Take a section XX at D and draw the free body diagram at D, as shown in fig.



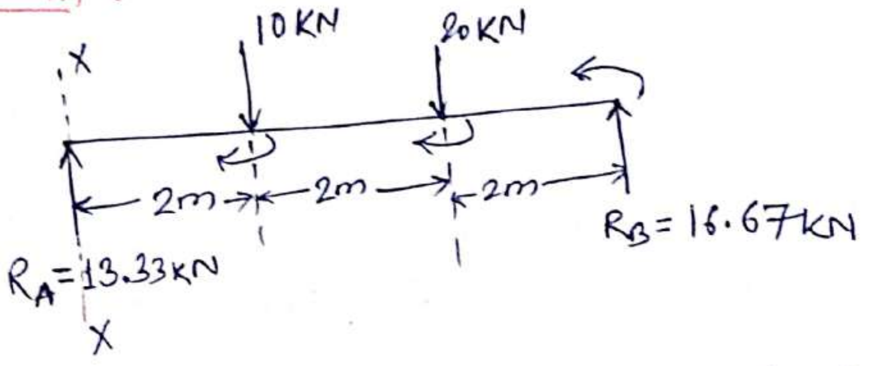
By considering the moments of all forces acting on the left side of D,

$$M_D = 13.33 \times 4 - 10 \times 2 = 33.32 \text{ KN-m}.$$

By Considering the moments of all forces acting on the right side of D,

$$M_D = 16.67 \times 2 = 33.34 \text{ KN-m}$$

To find M_A:



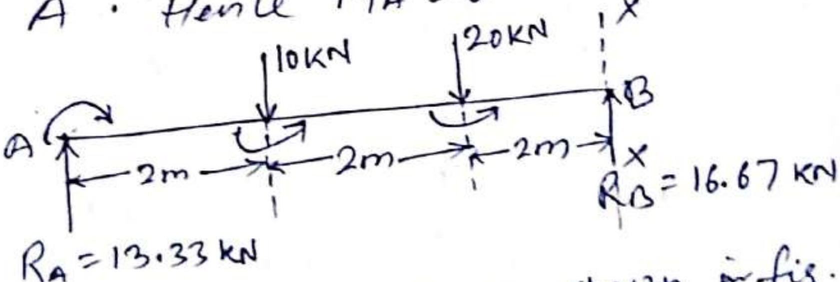
Take a section XX at A as shown in fig.

By Considering all the moments of all the forces acting right side of A.

$$M_A = -10 \times 2 - 20 \times 4 + 16.67 \times 6 = 0$$

Naturally, there is no force acting on left side of A. Hence $M_A = 0$

To find M_B:-

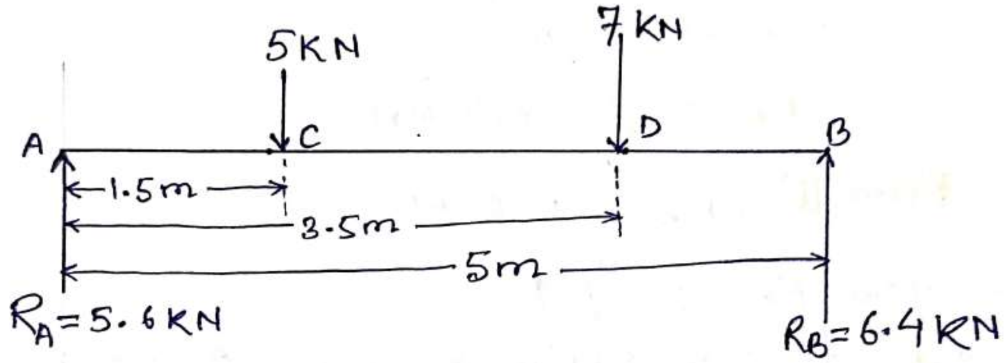


Take a section XX at B as shown in fig.

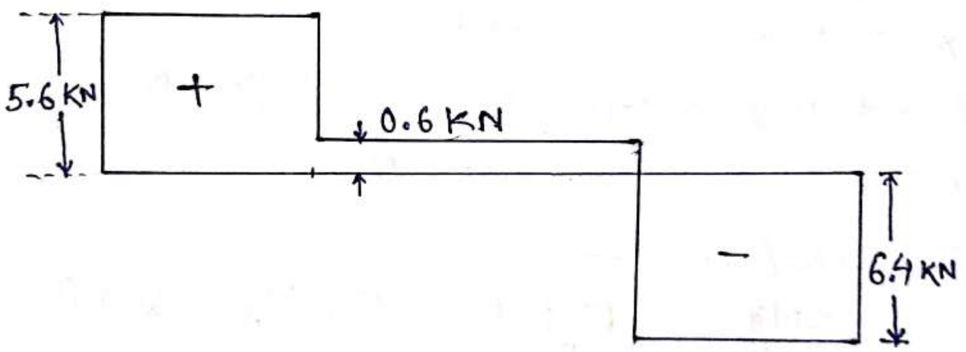
By considering all moments of all forces acting on left side of B, $M_B = 13.33 \times 6 - 10 \times 4 - 20 \times 2$
$= 0$

Naturally, there is no force acting on the right side of B, $M_B = 0$.

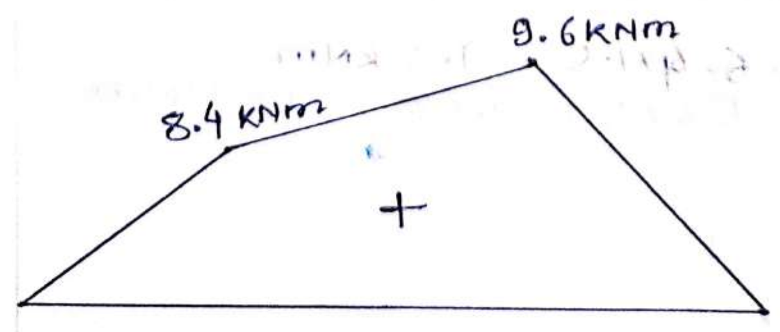
Q.1 A simply supported beam of span 5m carries two point loads of 5 KN and 7 KN at 1.5m and 3.5 m from the left hand support respectively. Draw S.F.D. and B.M.D. showing the important points/values.



(a) Beam



(b) S.F.D.



(c) B.M.D.

Reactions:-

$$\Sigma F_y = 0$$

$$R_A + R_B - 5 - 7 = 0$$

$$R_A + R_B = 12 \quad \text{——— (i)}$$

$$\Sigma M_A = 0$$

$$5\times1.5 + 7\times3.5 - R_B\times5 = 0$$

$$32 - 5R_B = 0$$

$$R_B = \frac{32}{5} = 6.4 \text{ KN}$$

From (i), $R_A = 5.6$ KN.

S.F. Calculations:-

S.F. at any section between A and C,

$F_x = +R_A = 5.6$ KN

S.F. at any section between C and D,

$F_x = 5.6 - 5 = 0.6$ KN

S.F. at any section between D and B,

$F_x = 5.6 - 5 - 7 = -6.4$ KN

B.M. Calculations:-

At simply supported ends, $M_A = M_B = 0$

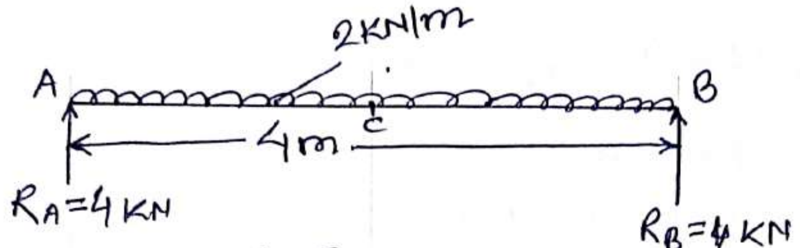
By considering the forces acting on left of C,

$M_C = 5.6\times1.5 = 8.4$ KN.m

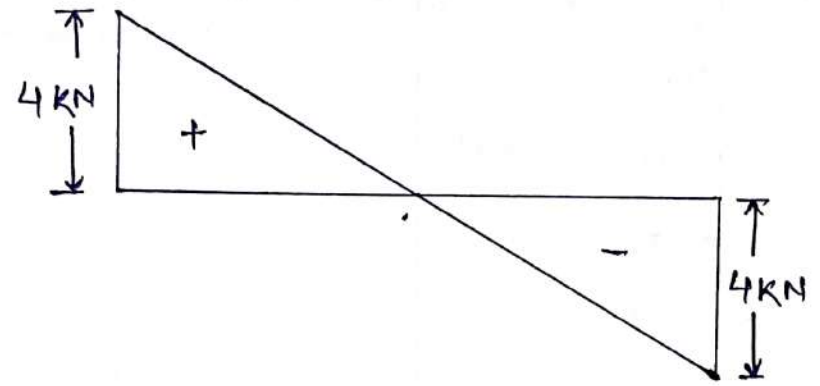
By considering the forces acting on left of D,

$M_D = 5.6\times3.5 - 5\times2 = 9.6$ KNm

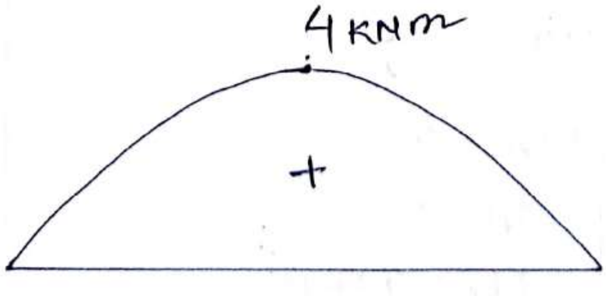
Q2. A simply supported beam of span 4 m carries a u.d.l. of 2 KN/m over 4m length. Draw S.F. and B.M. diagrams.



(a) Beam



(b) S.F.D.



(c) B.M.D.

Reactions:- $\Sigma F_Y = 0$

$$R_A + R_B - 2\times4 = 0$$

$$R_A + R_B = 8 \quad \text{——(i)}$$

$$\Sigma M_A = 0 \; [\circlearrowright^{+}, \circlearrowleft^{-}]$$

$$2\times4\times\frac{4}{2} - R_B\times4 = 0$$

$$R_B = 4\text{ KN}$$

from (i), $R_A = 4\text{ KN}$

S.F. Calculations:-

S.F. at any section between A and B at a distance x from A.

$$F_x = 4 - 2x$$

At $x=0$, $F_A = 4\text{ KN}$

At $x=4$, $F_B = -4\text{ KN}$

At $x=2$, $F_C = 0$ (At the centre of beam)

In this case, S.F.D. is an inclined straight line.

B.M. Calculations:-

~~At simply supported beam ends, $M_A = M_B = 0$~~

B.M. at any section between A & B at a distance x from A:

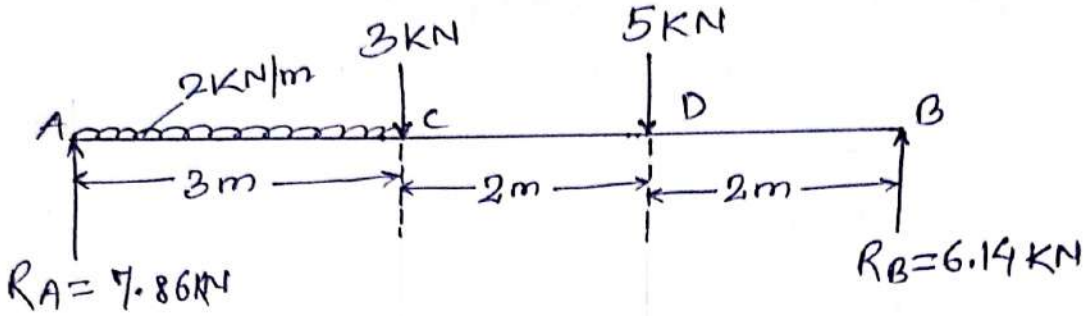
$$M_x = 4x - 2\times x\times\frac{x}{2}$$

At $x=0$, $M_A = 0$

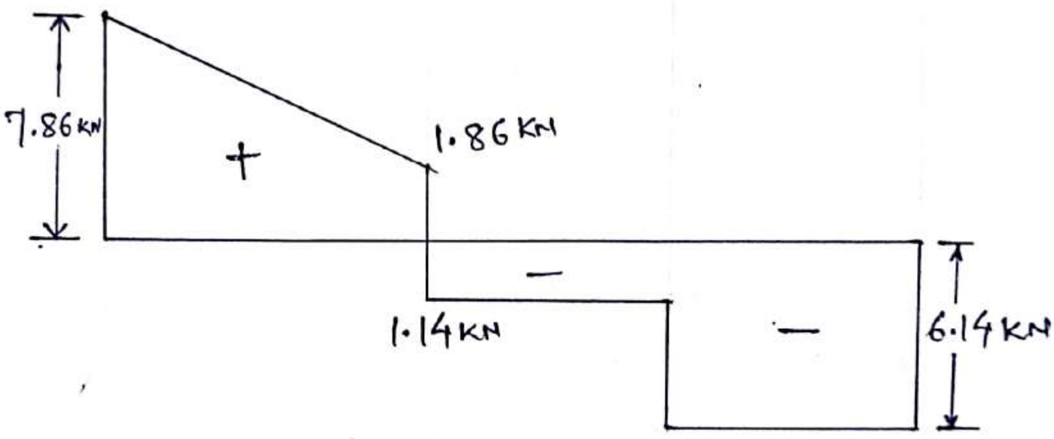
At $x=4$, $M_B = 0$

At $x=2$, $M_C = 4\text{ KNm}$

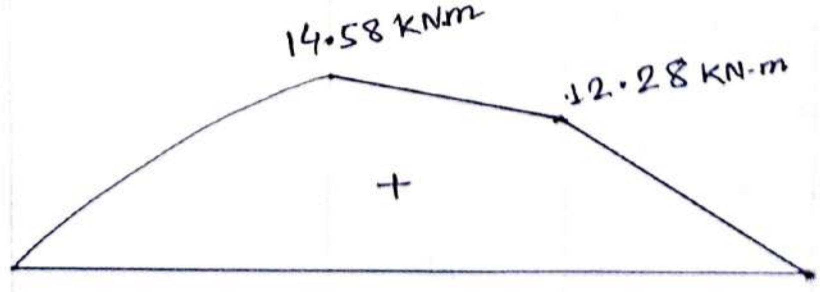
Q3. Draw S.F.D. and B.M.D. for the beam as shown in fig.



(a) Beam



(b) S.F.D.



(c) B.M.D.

Reactions:- $\Sigma F_Y = 0$

$$R_A - 2\times3 - 3 - 5 + R_B = 0$$

$$R_A + R_B = 14 \quad \text{——— (i)}$$

$\Sigma M_A = 0,$

$$2\times3\times\frac{3}{2} + 3\times3 + 5\times5 - R_B\times7 = 0$$

$$43 - 7R_B = 0$$

$$R_B = \frac{43}{7} = 6.14\ \text{KN}$$

from (1), $R_A = 14 - 6.14 = 7.86\ \text{KN}$

S.F. Calculation:-

S.F. at any section between A and C at a distance 'x' from A,

$$F_x = 7.86 - 2\times x$$

At $x = 0$, $F_A = 7.86$ KN

At $x = 3$ m, $F_C = 7.86 - 2\times3 = 1.86$ KN

S.F. at any section between C and D at a distance 'x' from A,

$$F_x = 7.86 - 2\times3 - 3$$

At $x = 3$, $F_C = -1.14$ KN

~~At x = 5, F_D = -5.14 KN~~

S.F. at any section in DB,

$$F_x = 7.86 - 2\times3 - 3 - 5$$

$$= -6.14\ \text{KN}$$

B.M. Calculations:-

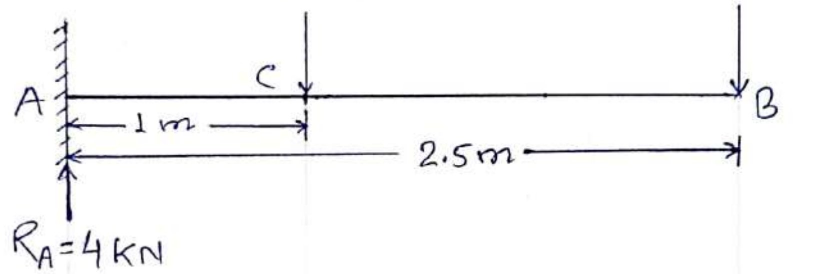
At simply support ends, $M_A = M_B = 0$

$$M_C = 7.86\times3 - 2\times3\times\frac{3}{2} = 14.58\ \text{KN·m}$$

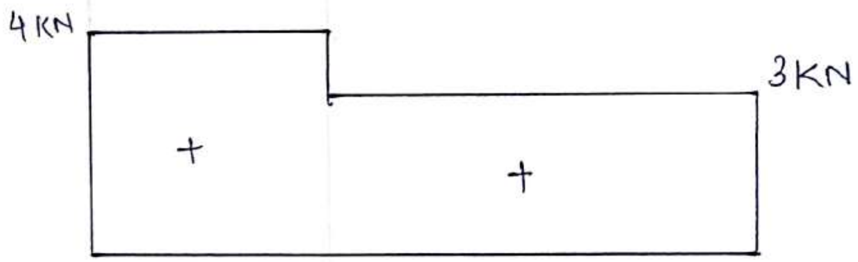
$$M_D = 7.86\times5 - 2\times3\times3.5 - 3\times2 = 12.30\ \text{KN·m}$$

Cantilever Beam

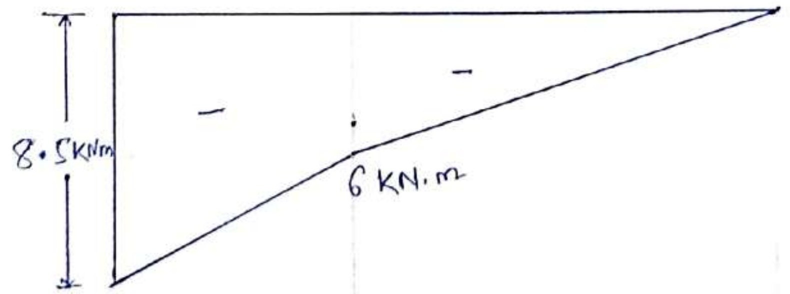
Q1. A Cantilever beam of span 2.5 m carries two point loads 1 KN and 3 KN at 1 m, 2.5 m from the fixed end. Draw S.F.D. and B.M.D.



(a) Beam



(b) S.F.D.



(c) B.M.D.

9

S.F. Calculation:-

S.F. at any section between C and B, $F_x = +3$ KN

S.F. at any section between A and C, $F_x = 3+1 = 4$ KN

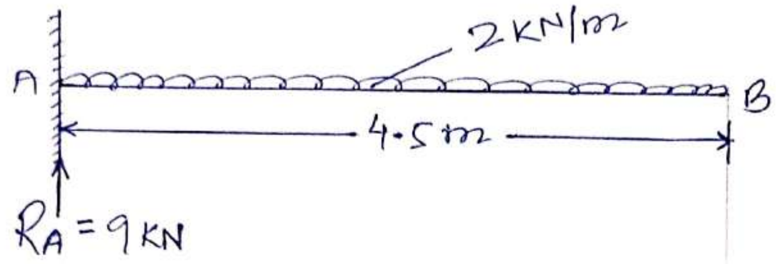
B.M. Calculation:

At the free end B, $M_B = 0$

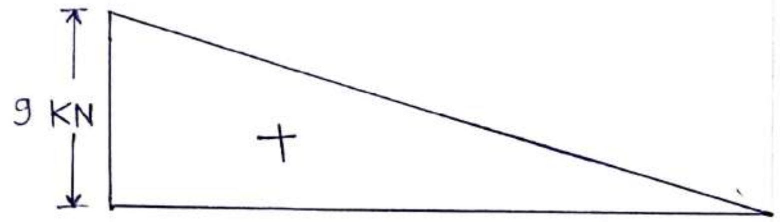
$M_C = -3 \times 2 = -6$ KN·m [minus sign due to clockwise moment to the right i.e. hogging]

$M_A = -3 \times 2.5 - 1 \times 1 = -8.5$ KN·m

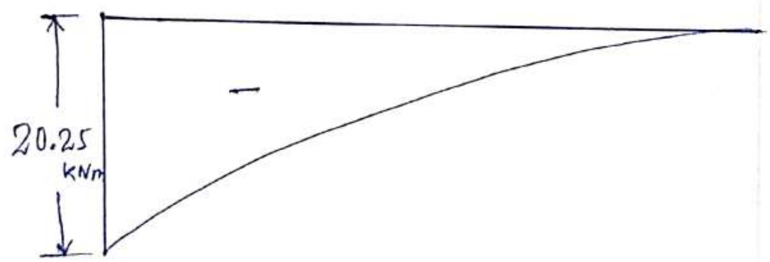
Q2: Draw S.F. and B.M. diagrams for the beam shown in figure.



ⓐ Beam



(b) S.F.D.



(c) B.M.D.

Reactions:-

$\Sigma F_Y = 0$

$R_A - 2 \times 4.5 = 0$

$R_A = 9$ KN.m

S.F. calculations:-

S.F. at any section between A and B at a distance x from A.

$$F_x = 9 - 2x$$

At $x = 0$, $F_A = 9$ KN

At $x = 4.5$, $F_B = 0$

At $x = \frac{4.5}{2}$, $F_C = 4.5$ KN

B.M. Calculations :-

B.M. at any Section between A and B at a distance x from B.

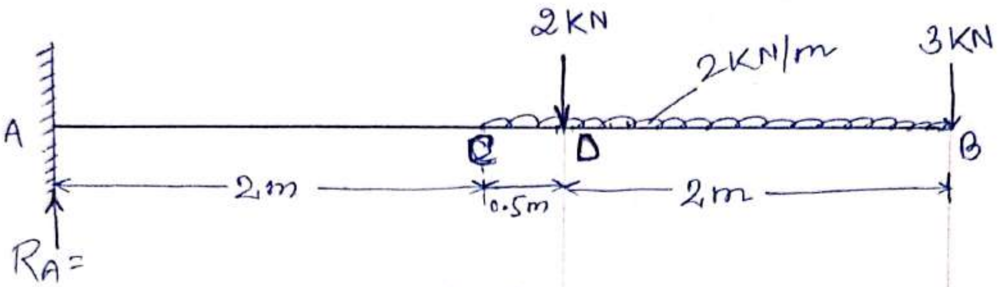
$$\cancel{M_x = 9x - 2 \times x \times \frac{x}{2}}$$

$$M_x = -2 \times x \times \frac{x}{2}$$

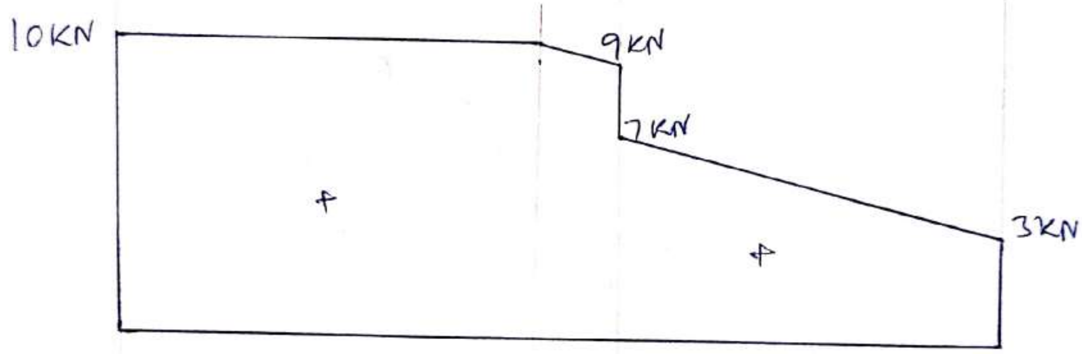
At B, $x = 0$, $M_B = 0$

At A, $x = 4.5$, $M_A = -20.25$ KN.m

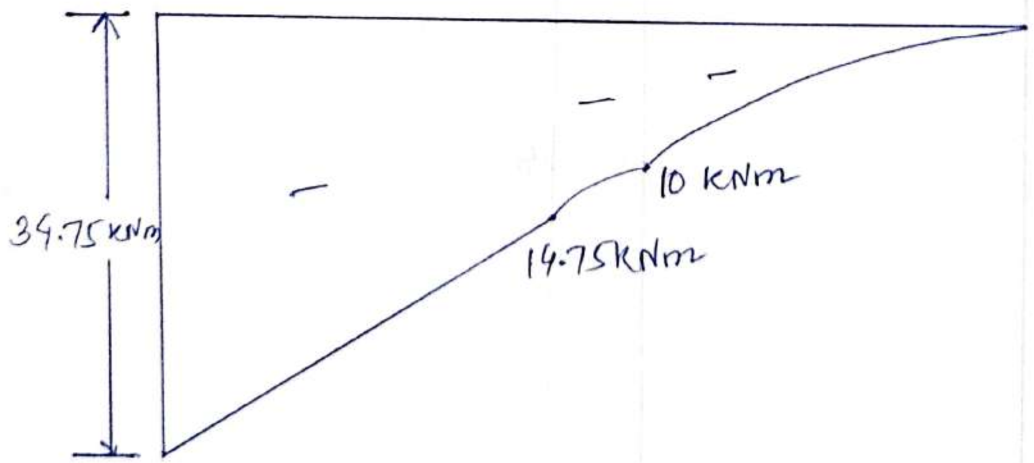
Q3. Draw S.F. and B.M. diagrams for the beam shown in fig.



(a) Beam



(b) S.F.D.



(c) B.M.D.

Reactions:-

$\Sigma F_Y = 0$

$R_A - 2 - 3 - 2\times2.5 = 0$

$R_A = 10$ kN.

S.F. Calculations:-

S.F. just to the left of B, $F_B = 3$ kN

S.F. just to the right of D, $F_{D_R} = 3 + 2\times2 = 7$ kN

S.F. just to the left of D, $F_{D_L} = 3 + 2\times2 + 2 = 9$ kN

S.F. just to the right of C, $F_{C_R} = 3 + 2\times2 + 2\times2\times0.5 = 10$ kN

Since there is no load between A and C, 10 kN remains constant from A to C. i.e. $F_A = 10$ kN.

B.M. calculations:-

$M_B = 0$

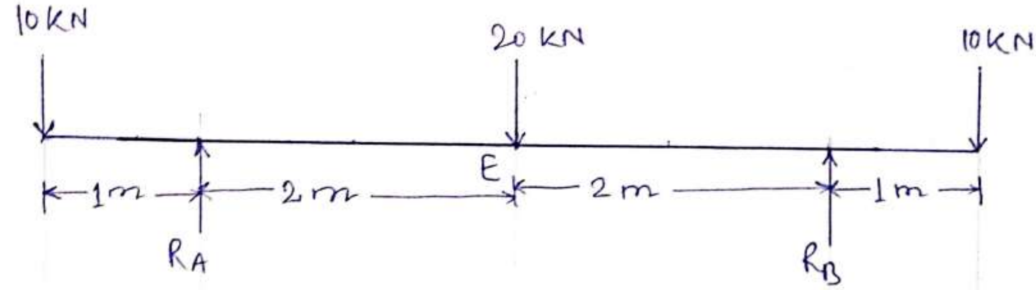
$M_D = -3\times2 - 2\times2\times\frac{2}{2} = -10$ kNm

$M_C = -3\times2.5 - 2\times2.5\times\frac{2.5}{2} - 2\times0.5 = -14.75$ kNm.

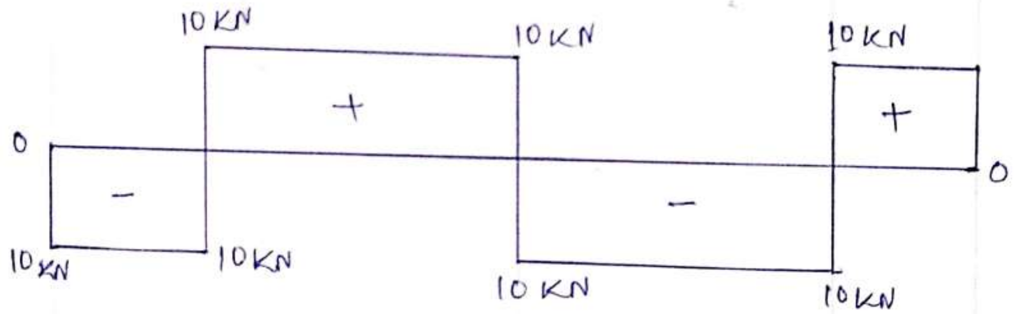
$M_A = -3\times4.5 - 2\times2.5\times\left(\frac{2.5}{2} + 2\right) - 2\times2.5 = -34.75$ kNm

Overhanging Beams

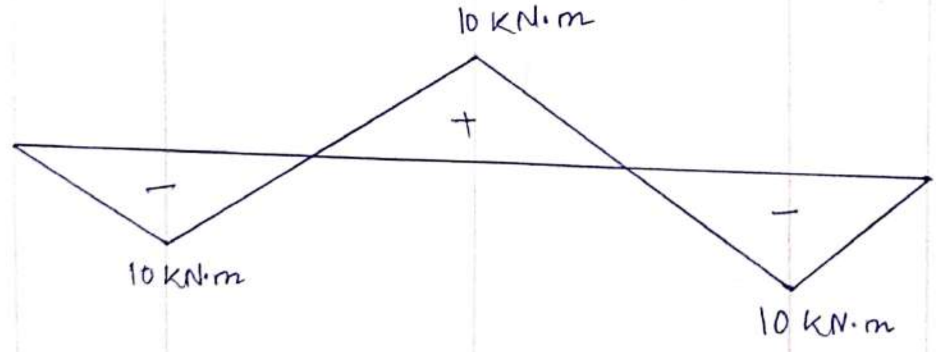
Q.1 A simply supported beam having equal overhangs on both sides and carrying point loads is shown in Fig. Draw S.F. and B.M. diagrams.



(a) Beam



(b) S.F.D.



(c) B.M.D.

Q2

Reactions:-

$$\Sigma F_Y = 0$$

$$R_A + R_B - 10 - 20 - 10 = 0$$

$$R_A + R_B = 40 \quad \text{--- (i)}$$

$$\Sigma M_A = 0, \; [\circlearrowright +, \circlearrowleft -]$$

$$-10 \times 1 + 20 \times 2 + 10 \times 5 - R_B \times 4 = 0$$

$$R_B = 20 \text{ KN}$$

$$R_A = 20 \text{ KN} \; [\text{from (i)}]$$

S.F. Calculations:-

S.F. at any section in C and A $= -10$ KN

S.F. at any section in A and E $= -10 + R_A = -10 + 20 = +10$ kN

S.F. at any section in E and B $= -10 + R_A + 20 = -10 + 20 - 20 = -10$ KN

S.F. at any section in B and D $= -10 + R_A - 20 + R_B$

$= -10 + 20 - 20 + 20 = +10$ kN.

B.M. Calculations:-

$M_C = 0$

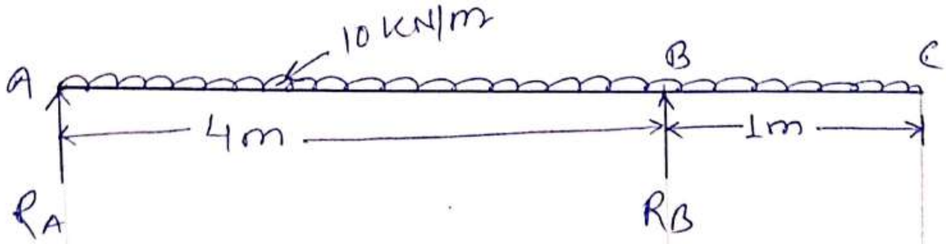
$M_A = -10 \times 1 = -10$ KN.m (Hogging)

$M_E = -10 \times 3 + R_A \times 2 = -10 \times 3 + 20 \times 2 = 10$ kNm (sagging)

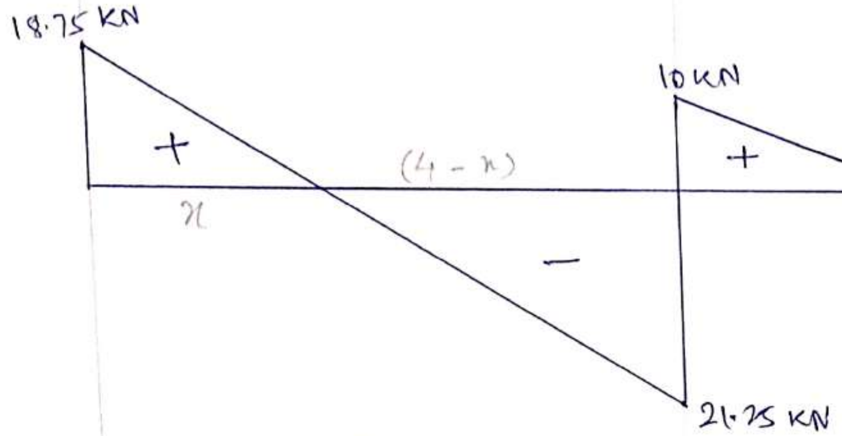
$M_B = -10 \times 1 = -10$ KN.m (Hogging)

$M_D = 0$.

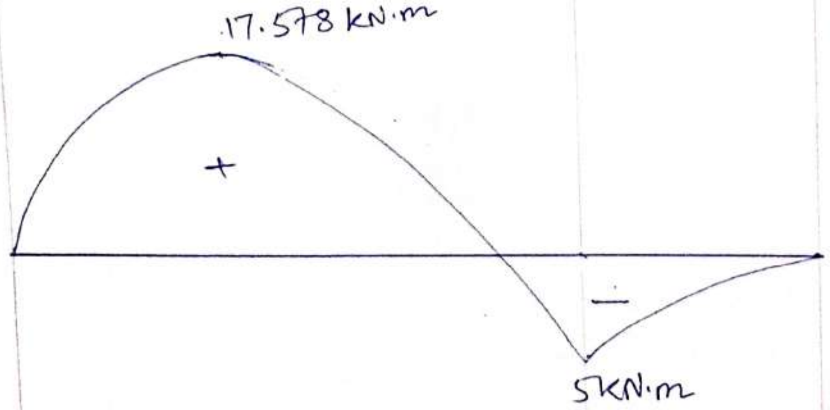
Q2 A simply supported beam having an overhang at one side carry u.d.l. of intensity 10 KN/m as shown in Fig. Draw S.F. and B.M. diagrams for the beam.



(a) Beam



(b) S.F.D.



(c) B.M.D.

Support reactions:-

$$\Sigma F_y = 0$$

$$R_A + R_B - 50 = 0$$

$$R_A + R_B = 50 \quad \text{—— (i)}$$

$$\Sigma M_A = 0, \quad R_B \times 4 - 10 \times 5 \times \frac{5}{2} = 0$$

$$R_B = 31.25 \text{ kN}.$$

$$R_A = 18.75 \text{ kN (from (i))}$$

S.F. Calculations:-

$$F_A = +R_A = 18.75 \text{ kN}$$

$$F_{B_L} = 18.75 - 10 \times 4 = -21.25 \text{ kN}$$

$$F_{B_R} = 18.75 - 10 \times 4 + 31.25 = 10 \text{ kN}$$

$$F_C = 18.75 - 10 \times 4 + 31.25 - 10 \times 1 = 0$$

B.M. Calculations:-

At simple support A, $M_A = 0$

$$M_B = -10 \times 1 \times \frac{1}{2} = -5 \text{ kN.m}.$$

Point of zero S.F. (Location of maximum B.M.)

From S.F.D., $\dfrac{18.75}{x} = \dfrac{21.25}{4 - x}$ [Similar Δ property)

$$x = 1.875 \text{ m}$$

$$M_{max} = M_{1.875} = R_A \times 1.875 - 10 \times 1.875 \times \frac{1.875}{2}$$

$$= 17.578 \text{ kN.m}$$

Point of Contraflexure:-

Let the point of contraflexure 'E' is at a distance x from A.

$$M_x = 18.75 \times x - 10 \times x \times \frac{x}{2} = 0$$

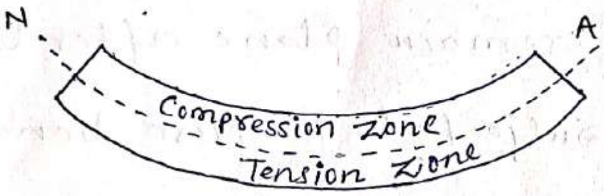
$$x = 3.75 \text{ m from A}.$$

Bending Moment:- Bending moment at any cross-section of the beam is the algebraic sum of the moments of all the forces acting on the right or left side of the section.

Bending Stress:- The stresses induced to resist the bending moment are called bending stresses.

Shear stress:- The stresses induced to resist the shear force are called shear stresses.
It is also called shearing stresses.

Neutral Axis:-



The fibres in the dotted centroidal layer are neither shortened nor elongated.
This centroidal layer which do not undergo any elongation or compression is called neutral layer or neutral surface.
The intersection of the neutral layer with any normal cross section of a beam is called neutral axis (N.A.).

Assumptions in the theory of bending

1. The material of the beam is homogeneous and isotropic. i.e. the beam is made up of same material throughout and it has the same elastic properties in all the directions.
2. The beam is straight before loading and is of uniform cross-section throughout.
3. The beam material is stressed within its elastic limit and thus obeys Hooke's law.
4. The transverse sections which were plane before bending remain plane after bending.
5. The beam is subjected to pure bending.
6. The value of modulus of elasticity (E) is same, for the fibres of the beam under compression or under tension.
7. Each layer of the beam is free to expand or contract independently of the layer above or below it.

Equation of bending

$$\boxed{\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}}$$

Where, M = Moment of resistance

I = Moment of inertia of the section about neutral axis

σ = bending stress

y = Distance of surface from neutral axis.

E = Modulus of elasticity

R = Radius of curvature of Neutral axis

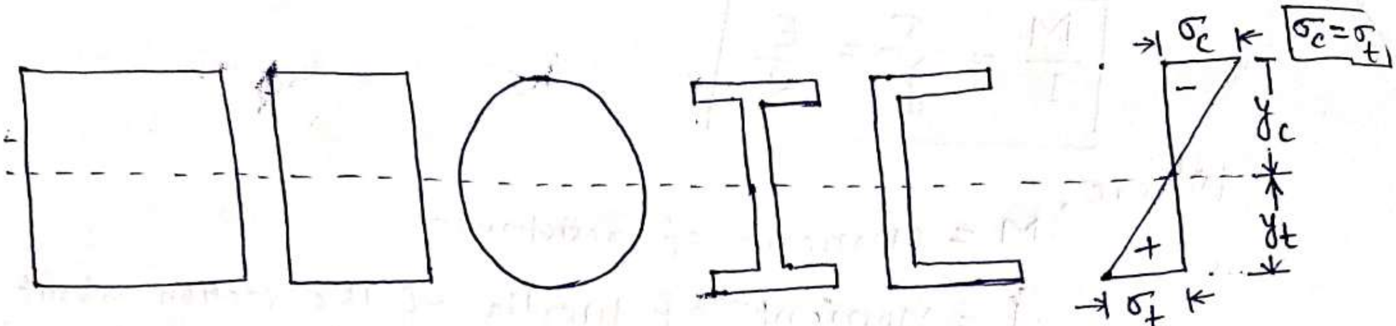
The above equation is known bending equation. It is also called Flexural formula.

Moment of resistance:-

Moment of resistance of the beam is the moment of couple formed by the total compressive force acting at the c.g. of the compressive stress diagram and the total tensile force acting at the c.g. of the tensile stress diagram. It resists the external bending moment. ~~In~~

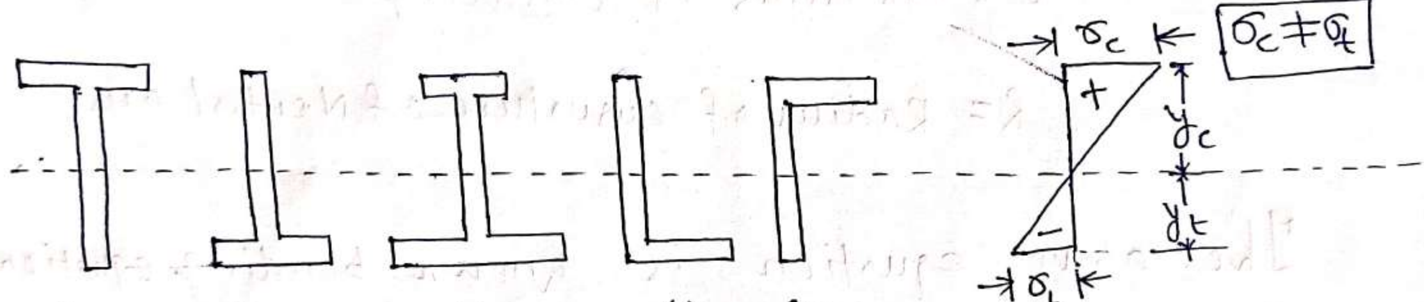
In the equilibrium condition, the moment of resistance must be equal to the external bending moment, i.e. $M_r = M$.

Bending stress distribution diagrams for symmetrical section



(a) Square (b) Rectangular (c) Circular (d) I (e) Channel (f) Bending stress diagram

Bending stress distribution diagrams for asymmetrical sections



(a) T (b) Inverted T (c) Unsymmetrical I (d) Angle (e) Angle (f) Bending stress diagram

Section Modulus :-

It is the ratio of moment of Inertia of the section about the neutral axis and the distance of the most extreme fibre from the neutral axis.

→ It is denoted by 'Z'.

→ It's S.I. unit is m^3

We know that from flexural formula,

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$M = \sigma\left(\frac{I}{y}\right).$$

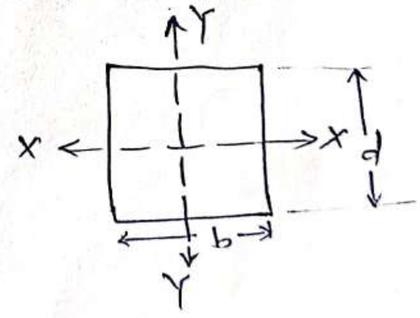
$$M = \sigma Z$$

where, $Z = \frac{I}{y}$ is called as section modulus.

① Section Modulus of rectangular section

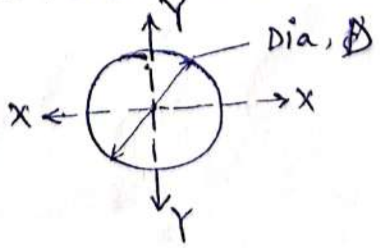
$$Z_{XX} = \frac{bd^2}{6}$$

$$Z_{YY} = \frac{db^2}{6}$$



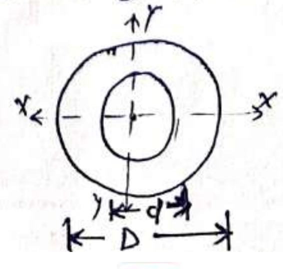
② Section Modulus of Solid circular section

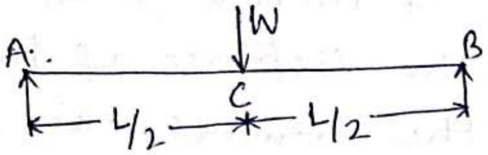
$$Z_{XX} = Z_{YY} = \frac{\pi}{32} D^3$$



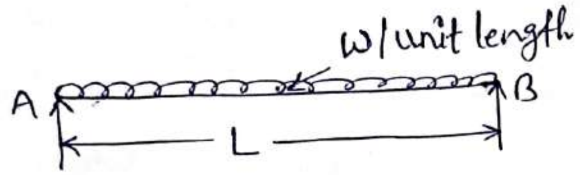
③ Section Modulus of hollow circular Section

$$Z_{XX} = Z_{YY} = \frac{\pi}{32}\left[\frac{(D^4 - d^4)}{D}\right]$$

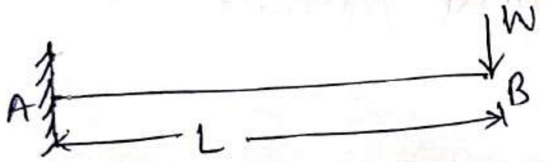


(1) 

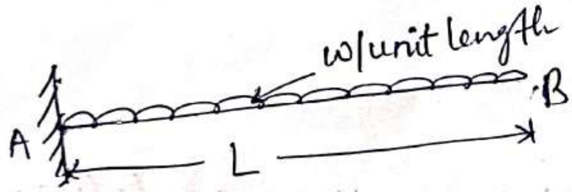
$$M_C = \frac{WL}{4}$$

(2) 

$$M_C = \frac{wL^2}{8}$$

(3) 

$$M_A = WL$$

(4) 

$$M_A = \frac{wL^2}{2}$$

Shear stress equation and the meaning of symbols used

→ In a simply supported beam subjected to some loading,

$$q = \frac{S A \bar{y}}{I b}$$

Where,

q = Intensity of shear stress induced in a layer at a distance y from the N.A.

A = Area of beam above the layer under consideration

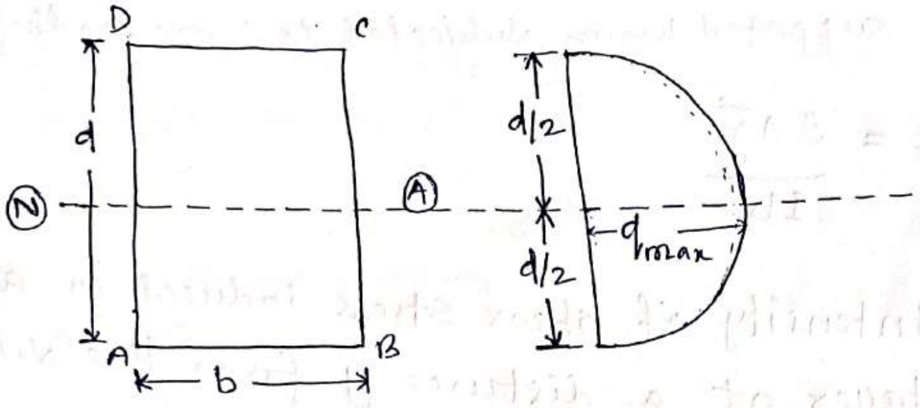
$\bar{y}$ = Distance of the c.g. of the area considered from the neutral axis

$A\bar{y}$ = Moment of the area above the layer considered about the neutral axis

I = Moment of inertia of the whole cross section about the neutral axis = I_{NA}.

b = Width of the section at a distance y from the neutral axis (N.A.)

Shear stress distribution for a rectangular section



(a) Cross section

(b) Shear stress distribution

M.I. of rectangular section about the N.A. $= \frac{bd^3}{12}$

Average shear stress $= \frac{\text{Shear force}}{\text{Area}}$

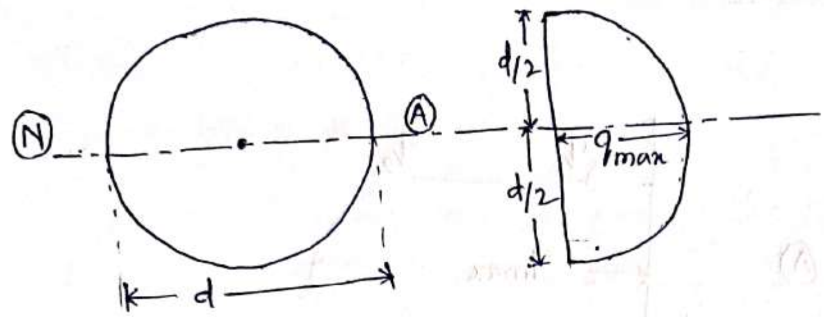
$$\boxed{q_{av} = \frac{S}{bd}}$$

Maximum shear stress $= 1.5\left(\frac{S}{bd}\right)$

$$\boxed{q_{max} = 1.5\, q_{av}}$$

Note:- for a rectangular section, the maximum shear stress is 1.5 times the average shear stress.

Shear stress distribution for a circular section



(a) Cross-section

(b) Shear stress distribution

M.I. of circular section about N.A. $= \frac{\pi}{64} d^4$

Average shear stress $= \frac{\text{Shear force}}{\text{Area}}$

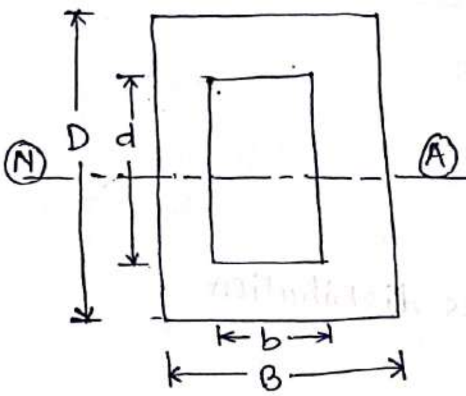
$$\boxed{q_{av} = \frac{S}{\frac{\pi}{4} d^2}}$$

Maximum shear stress $= \frac{4}{3} \frac{S}{\frac{\pi}{4} d^2}$

$$\boxed{q_{max} = \frac{4}{3} q_{av}}$$

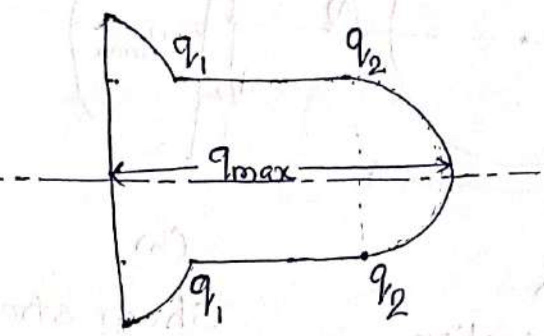
Note :- For a circular section, the maximum shear stress is $\frac{4}{3}$ times the average shear stress.

Shear stress distribution diagram for a hollow rectangular section



(a)

Cross-Section



(b)

Shear stress distribution

M.I. of hollow rectangular section about N.A. = $I = \frac{1}{12}(BD^3 - bd^3)$

Average shear stress = $\frac{\text{Shear force}}{\text{Area}}$

$$\boxed{q_{av} = \frac{S}{BD - bd}}$$

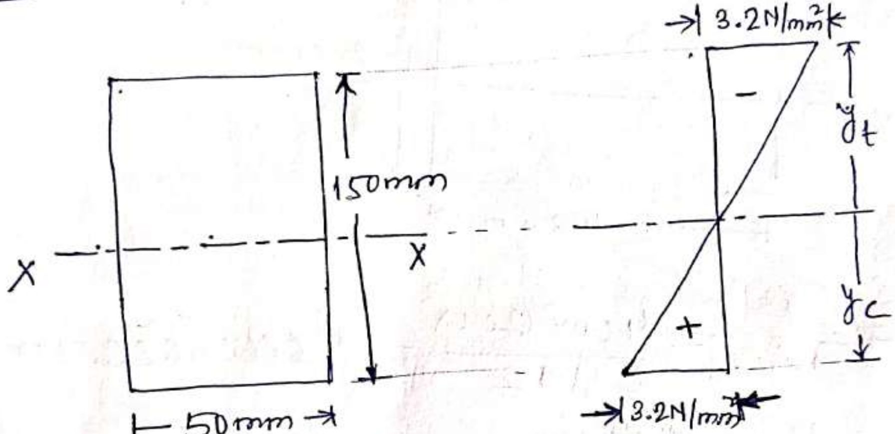
Maximum shear stress =

Q. Determine the maximum bending stress developed in a beam of rectangular cross-section 50 mm × 150 mm when a bending moment of 600 N·m is applied about x–x-axis.

Soln:– Given: $b = 50\ mm$, $d = 150\ mm$,

$M = 600\ N \cdot m = 600 \times 10^3\ N \cdot mm$

Find: σ



(i) Section

(ii) Bending stress distribution

$$I_{xx} = \frac{bd^3}{12} = \frac{50 \times 150^3}{12} = 14062500\ mm^4$$

$$y = \frac{d}{2} = \frac{150}{2} = 75\ mm$$

We know that, $\frac{M}{I} = \frac{\sigma}{y}$

or, $\frac{600 \times 10^3}{14062500} = \frac{\sigma}{75}$

or, $\sigma = \frac{600 \times 10^3 \times 75}{14062500} = 3.2\ N/mm^2$

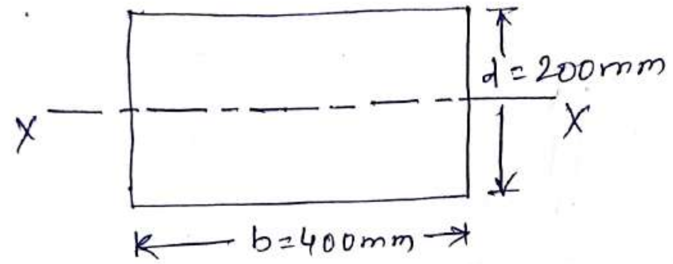
$\therefore \boxed{\sigma = 3.2\ N/mm^2}$

Scanned with OKEN Scanner

Q. A rectangular beam of 400mm x 200mm size is of wood material. If the permissible bending stress in wood is 2 N/mm², calculate the moment of resistance of beam.

Soln:- $b = 400\ mm$, $d = 200\ mm$

$\sigma = 2 N/mm^2$



$$I_{xx} = \frac{bd^3}{12} = \frac{400 \times (200)^3}{12} = 266666666.7\ mm^4$$

$$y = \frac{d}{2} = \frac{200}{2} = 100\ mm$$

Now, $\frac{M}{I} = \frac{\sigma}{y}$

$$\frac{M}{266666666.7} = \frac{2}{100}$$

$$M = \frac{2 \times 266666666.7}{100}$$

$$M = 5.33 \times 10^6\ N \cdot mm$$

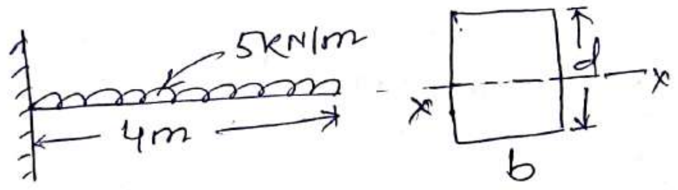
$$M = 5.33 \times 10^3\ KN \cdot mm$$

$$\boxed{M = 5.33 \times 10^3\ KN \cdot m}$$

Scanned with OKEN Scanner

Q. A cantilever beam of 4 m span carries a u.d.l. of 5 kN/m and permissible stress in the material of the beam is 5 N/mm². Design the section of beam if depth to width ratio is 2.

Soln:- Given: $L = 4m$, $\omega = 5 kN/m$, $\sigma = 5 N/mm^2$,
$= \cancel{5 N/mm}$
$d = 2b$



For a cantilever beam carrying u.d.l. over the entire span,

$$M = \frac{\omega L^2}{2} = \frac{5 \times 4^2}{2} = 40 kN\cdot m = 40 \times 10^6 N\cdot mm$$

$$I_{xx} = \frac{bd^3}{12} = \frac{b \times (2b)^3}{12} = \frac{8b^4}{12} = \frac{2}{3}b^4$$

$$y = \frac{d}{2} = \frac{2b}{2} = b$$

Using bending stress equation,

$$\frac{M}{I} = \frac{\sigma}{y}$$

or, $$\frac{40 \times 10^6}{\frac{2b^4}{3}} = \frac{5}{b}$$

or, $$b^3 = 12 \times 10^6$$

$$b = 228.94 \, mm \approx 230 \, mm$$
$$d = 2b = 2 \times 230 = 460 \, mm$$ Ans

Q. Find the section modulus of a rectangular section 200 mm x 500 mm about x-axis.

Solⁿ:- Given: $b = 200\,mm$, $d = 500\,mm$

Find : Z_{xx}

$$I_{xx} = \frac{bd^3}{12}, \quad y_{max} = \frac{d}{2}$$

$$Z_{xx} = \frac{I_{xx}}{y_{max}} = \frac{bd^3/12}{d/2} = \frac{bd^2}{6} = \frac{200 \times (500)^2}{6}$$

$$\boxed{Z_{xx} = 8.33 \times 10^6\ mm^3} \quad \text{Ans}$$

Q. A simply supported beam of span 3 m carries a u.d.l. of 1000 N/m throughout the span. Calculate the modulus of section if the permissible bending stress for the material is 9 MPa.

Solⁿ:- Given: $L = 3\,m$, $\omega = 1000\,N/m$, $\sigma = 9\,MPa = 9\,N/mm^2$

Find:- Z

For a simply supported beam carrying u.d.l. over the entire span,

$$M_{max} = \frac{\omega L^2}{8} = \frac{1000 \times 3^2}{8} = 1.125 \times 10^3\ N.m$$
$$= 1.125 \times 10^6\ N.mm$$

Using the bending stress equation,

$$\frac{M}{I} = \frac{\sigma}{y}$$

or, $$\frac{1.125 \times 10^6}{I} = \frac{9}{y}$$

or $$\frac{I}{y} = \frac{1.125 \times 10^6}{9} = 0.125 \times 10^6\ mm^3$$

or $$Z = I/y = 0.125 \times 10^6\ mm^3 \quad \text{Ans}$$

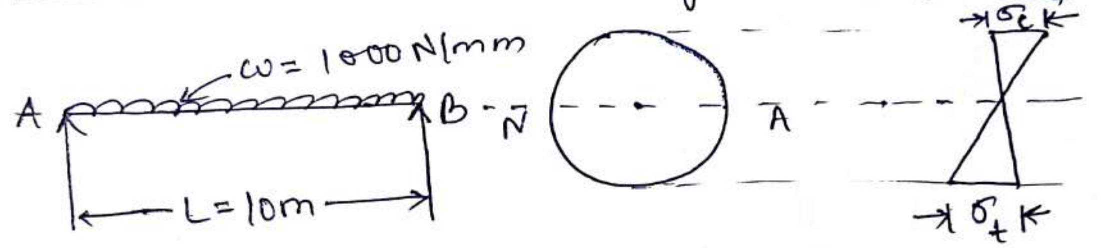
Q. A circular beam of 120 mm diameter is simply supported over a span of 10m and carries a u.d.l. of 1000 N/m. Find the maximum bending stress produced.

Soln:- Given: diameter, $d = 120$ mm

Length of beam, $L = 10m = 10 \times 10^3$ mm

u.d.l., $w = 1000$ N/m $= \frac{1000 N}{10^3 mm} = 1$ N/mm

Find:- Maximum bending stress, $\sigma = ?$



S.S. beam

Cross-Section

Bending stress diagram

For a simply supported beam carrying u.d.l. over the span of beam,

Bending moment, $M_{max} = \frac{wL^2}{8}$

$= \frac{1 \times (10 \times 10^3)^2}{8} = 125 \times 10^5$ N-mm

Moment of Inertia of a circular cross-section,

$$I = \frac{\pi}{64} d^4 = \frac{\pi}{64} \times (120)^4 = 10178760.2 \text{ mm}^4$$

$y = \frac{d}{2} = \frac{120}{2} = 60$ mm

Using bending equation, $\frac{M}{I} = \frac{\sigma}{y}$

or $\frac{125 \times 10^5}{10178760.2} = \frac{\sigma}{60} \Rightarrow \sigma = 73.68$ N/mm^2 Ans

Q. A cantilever beam of span 6.5 m is having cross-section of 400 mm wide and 700 mm deep. If the bending stress is not allowed to exceed 280 N/mm², calculate the magnitude of point load which can be applied at the free end of this cantilever beam.

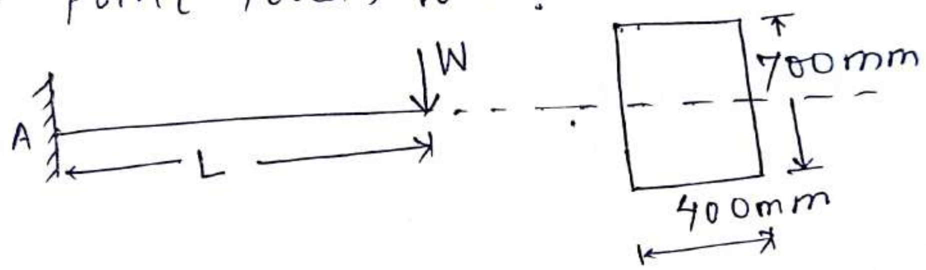
Soln:- Given: Length, $L = 6.5\,m = 6.5 \times 10^3\,mm$

Width, $b = 400\,mm$

Depth, $d = 700\,mm$

Bending stress, $\sigma = 280\,N/mm^2$

Find:- point load, $W = ?$



A cantilever beam of span L carrying a point load W, Bending Moment, $M = WL = W \times 6.5 \times 10^3$

Moment of Inertia, $I = \frac{bd^3}{12} = \frac{400 \times (700)^3}{12} = 1.143 \times 10^{10}\,mm^4$

$y = \frac{d}{2} = \frac{700}{2} = 350\,mm$.

Using bending equation,

$$\frac{M}{I} = \frac{\sigma}{y}$$

or, $$\frac{W \times 6.5 \times 10^3}{1.143 \times 10^{10}} = \frac{280}{350}$$

or, $W = 1406769.23\,N$ Ans

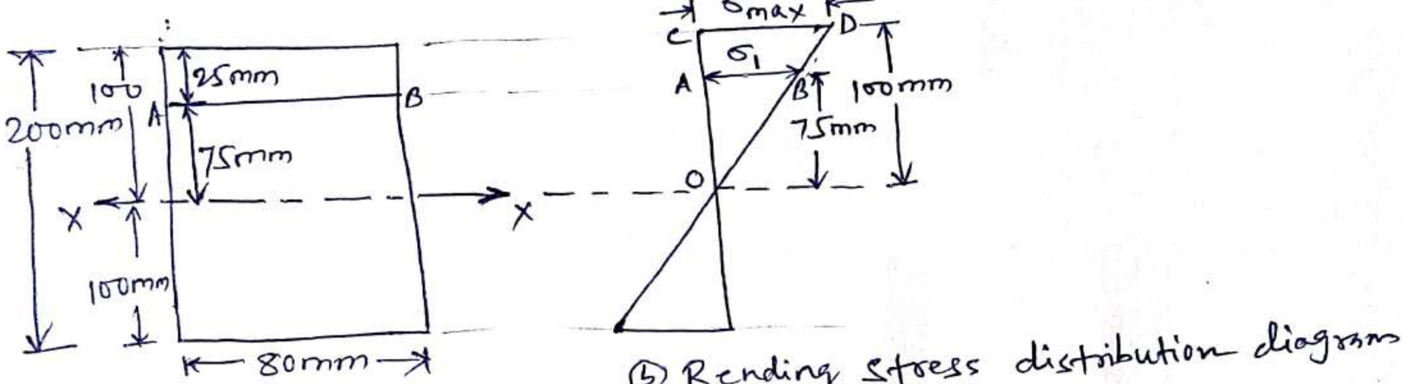
Q. Find the bending stress at 25 mm below the top edge of rectangular section 80 mm wide and 200 mm deep, if maximum bending moment is 4 kN·m.

Solution :- Given :- width, $b = 80 mm$

depth, $d = 200 mm$

$M_{max} = 4 kN \cdot m = 4 \times 10^3 N \cdot m = 4 \times 10^3 \times 10^3 N\text{-}mm$

$= 4 \times 10^6 N\text{-}mm$



(a) Section

(b) Bending stress distribution diagram

using bending diagram,

$$\frac{M}{I_{xx}} = \frac{\sigma_{max}}{y_{max}}$$

or $$\frac{4 \times 10^6}{\frac{80 \times 200^3}{12}} = \frac{\sigma_{max}}{\frac{200}{2}}$$

or $\sigma_{max} = 7.5 N/mm^2$

Let σ_1 = bending stress in layer AB which is 25 mm below the top edge.

From similar triangles ΔOCD & ΔOAB

$$\frac{CD}{OC} = \frac{AB}{OA}$$

or, $$\frac{\sigma_{max}}{100} = \frac{\sigma_1}{75}$$

or $$\frac{7.5}{100} = \frac{\sigma_1}{75}$$

or, $\sigma_1 = 5.625 N/mm^2$ Ans

Q. A steel strip 40 mm wide and 6 mm thick is subjected to end couples 20 N-m. Find the radius of curvature of the bent up strip, if $E = 2\times10^5$ MPa.

Solution:- Given: width, $b = 40$ mm

Thick, $d = 6$ mm

Couple or Bending Moment, $M = 20$ Nm $= 20\times10^3$ N-mm

$E = 2\times10^5$ MPa $= 2\times10^5$ N/mm²

Find:- Radius of curvature, $R = ?$

Using bending equation,

$$\frac{M}{I} = \frac{E}{R}$$

or $$\frac{20\times10^3}{\frac{40\times6^3}{12}} = \frac{2\times10^5}{R}$$

or $R = 7200$ mm $= 7.2$ m Ans

Q. A beam having modulus of elasticity of 2.8×10^5 N/mm² is bent with radius of curvature of 28 m under the effect of bending moment of 5000 N.mm. Calculate the moment of inertia of the cross-section of the beam.

<u>solution</u>:- Given:- $E = 2.8 \times 10^5$ N/mm²

$R = 28\,m = 28 \times 10^3$ mm

$M = 5000$ N-mm

Find:-, Moment of inertia, I.

using bending equation,

$$\frac{M}{I} = \frac{E}{R}$$

or, $$\frac{5000}{I} = \frac{2.8 \times 10^5}{28 \times 10^3}$$

or $I = 500$ mm⁴ Ans

CH-8: Finite Element Method [FEM]

Introduction to Finite Element Methods (FEM):

The **Finite Element Method (FEM)** is a powerful computational technique used to solve complex problems in engineering and physics, particularly those involving structures, fluids, heat transfer, and electromagnetism. It is widely applied in various fields such as mechanical engineering, civil engineering, aerospace engineering, and biomedical engineering.

FEM allows for the approximation of solutions to partial differential equations (PDEs) by breaking down complex problems into simpler, smaller components.

FEM is a method for solving partial differential equations (PDEs) that arise from physical phenomena, such as heat conduction, fluid flow, and structural deformation. It involves breaking down a large, complex system into smaller, simpler parts called "finite elements." These elements are connected at points called "nodes." By approximating the solution within each element and assembling the solutions for all elements, FEM can provide an approximate solution to the original problem.

Methods employed in FEM- Steps in FEM

1. Problem Definition

FEM is typically used to solve boundary value problems, which involve finding the unknown behavior (e.g., displacement, temperature, or stress) of system under given conditions, subject to certain constraints. These problems can be described by:

- **Partial Differential Equations (PDEs):** These govern the behavior of physical systems, such as heat conduction, fluid flow, or structural deformations.
- **Boundary Conditions:** Conditions that specify the behavior of the system at its boundaries, like fixed supports or specified temperatures.
- **Initial Conditions (if time-dependent):** Describes the initial state of the system at the start of the analysis.

2. Discretization of the Domain

In FEM, the first step is to **discretize** the physical domain (the structure or material of interest). The domain is divided into small, simple shapes called **finite elements**, which can be triangles, quadrilaterals, tetrahedra, or hexahedra depending on the problem's dimensionality and geometry. These elements are connected at points known as **nodes**.

- **Element:** A small, simple subdomain that can be solved independently.
- **Node:** The points that define the corners of each element and where the solution (e.g., displacement, temperature) is computed.
- **Mesh:** The entire collection of elements and nodes that form the discretized domain.

3. Formulation of the Problem

Each element is modeled mathematically using a set of **shape functions** that interpolate the solution within the element. The solution is usually approximated by expressing the unknown variable (e.g., displacement or temperature) as a linear combination of shape functions.

- **Shape Functions:** These define how the solution behaves within each element.
- **Element Stiffness Matrix:** For structural problems, the stiffness matrix describes the relationship between the forces applied to the element and the displacements that result from those forces.
- **Global System of Equations:** The system of equations for all elements is assembled into a large system, typically written as:
  [K]{u}={F}
  Where,
  [K] is the global stiffness matrix,
  {u} is the vector of unknown displacements, and
  {F} is the vector of external forces or loads applied to the system.

4. Assembly of the Global System

The contribution of each element's stiffness matrix and force vector is assembled into a global system of equations. This involves summing up the local stiffness matrices and force vectors from all elements, taking into account how the elements share nodes.

5. Solution of the System

Once the global system of equations is established, it is typically solved using numerical techniques such as direct solvers (e.g., Gaussian elimination) or iterative methods (e.g., Conjugate Gradient). The result is a set of approximate values for the unknowns at the nodes (e.g., displacements, temperatures, etc.).

6. Post-Processing

After solving for the unknowns, the results are analyzed and interpreted. This might involve:

- **Visualization:** Graphical representations such as deformed shapes, stress distributions, or temperature gradients.
- **Validation:** Comparing the results with experimental data or known analytical solutions to ensure accuracy.
- **Refinement:** Refining the mesh (i.e., making the elements smaller) to improve accuracy if necessary.

Finite Element Method (FEM) - Concept of Discontinuity

FEM is a powerful numerical method used for solving complex engineering problems, especially those involving structures, fluids, and heat transfer. It divides a large problem into smaller, simpler parts called elements, which are connected at nodes. While FEM is highly versatile and effective, there are both advantages and disadvantages to using it, especially in cases involving discontinuities (such as cracks, holes, or sudden material property changes).

Advantages of FEM:

1. **Versatility**: FEM can be applied to a wide range of problems including static, dynamic, linear, and nonlinear analyses. It is used in solid mechanics, fluid dynamics, heat transfer, electromagnetic fields, and more.
2. **Complex Geometries**: FEM can handle complex geometries and boundary conditions, making it suitable for irregular shapes and structures.
3. **Discretization**: The domain of the problem is divided into smaller, manageable parts (elements), making the problem solvable.
4. **Adaptability to Discontinuities**: FEM is capable of addressing problems involving discontinuities such as cracks, holes, and material property changes by refining the mesh in those regions.
5. **Local Refinement**: In regions with high gradients (like discontinuities), FEM allows mesh refinement to capture the variations more accurately without needing to refine the entire model.
6. **Accurate Results**: The method provides accurate solutions, especially when higher-order elements are used and the mesh is refined sufficiently in critical areas.
7. **Support for Nonlinear Analysis**: FEM can handle nonlinear material behavior, large deformations, and boundary conditions, which is crucial for real-world applications.

Disadvantages of FEM:

1. **Computational Cost**: FEM simulations, especially for large models with complex geometries or when dealing with fine meshes, can be computationally expensive and time-consuming.
2. **Complexity of Setup**: Setting up a FEM model requires significant expertise, particularly when handling complex boundary conditions, material properties, or nonlinear behavior.
3. **Mesh Dependency**: The accuracy of the solution depends heavily on the mesh quality. Poor mesh choices can lead to inaccurate results, particularly near discontinuities.
4. **Handling Discontinuities**: While FEM can handle discontinuities, it is challenging to represent them perfectly without introducing errors, especially in regions with sharp gradients or stress concentrations.
5. **Convergence Issues**: Some problems, especially those with highly nonlinear or time-dependent behavior, may face convergence difficulties if not properly set up or if the mesh isn't refined enough.
6. **Post-Processing Requirements**: Interpretation of results requires sophisticated post-processing tools to extract meaningful engineering insights.

Limitations of FEM:

1. **Singularity at Discontinuities**: FEM can struggle with representing sharp discontinuities (like cracks or material interfaces) because these singularities often require very fine meshes to resolve, which can be computationally prohibitive.
2. **Stress Concentrations**: At points of discontinuity, stress concentrations may occur. FEM approximates these concentrations, but accurate representation often requires very dense meshes, which increases computational load.
3. **Material Property Changes**: Discontinuities in material properties (such as a sudden change in material composition or phase) require special handling, and errors may arise if the material interface is not modeled properly.
4. **Fracture Mechanics**: While FEM can model crack propagation, handling dynamic fracture (moving cracks or growing fractures) requires advanced techniques like cohesive zone models or extended finite element methods (XFEM), which can be complex to implement.
5. **Boundary Conditions**: Discontinuities at boundaries (e.g., a hole in a plate) may result in problems with boundary condition application or numerical artifacts if the boundary is not carefully defined.

Applications of FEM:

1. **Structural Analysis**: FEM is widely used in structural engineering to analyze stresses, strains, and deformations in buildings, bridges, and mechanical components.
2. **Heat Transfer**: It is used to model heat conduction, convection, and radiation, especially in complex systems with varying material properties and boundary conditions.
3. **Fluid Dynamics**: FEM is applied to solve fluid flow problems, particularly when there are changes in material properties or complex geometries.
4. **Acoustic Analysis**: FEM is used to study sound wave propagation, vibration, and noise control in various structures.
5. **Electromagnetic Fields**: FEM is used in electrical engineering to analyze fields in devices like antennas, capacitors, and magnetic materials.
6. **Fracture Mechanics**: FEM, especially with extensions like XFEM, is used to model crack growth and failure in materials, which is crucial in aerospace, automotive, and civil engineering.
7. **Biomechanics**: FEM is applied to simulate human tissues, bones, and organs under various forces for medical and prosthetic design.
8. **Geotechnical Engineering**: Used for soil-structure interaction problems, FEM is applied in the design of foundations, tunnels, and other geotechnical structures.

Chapter-9

Finite Element Analysis

Steps in Finite Element Analysis (FEA)

The general procedure for performing FEM involves the following steps:

1. **Preprocessing**:
   - Define the problem (governing equations and boundary conditions).
   - Discretize the domain into finite elements (mesh generation).
   - Choose an appropriate element type (e.g., 1D, 2D, or 3D elements).
   - Assign material properties and load conditions.
2. **Solution**:
   - Formulate the system of equations based on the weak form.
   - Assemble the global stiffness matrix.
   - Apply boundary conditions and solve the system of equations to get the unknown values (e.g., displacement, temperature).
3. **Postprocessing**:
   - Analyze the results (e.g., visualize the displacement, stress, or temperature distribution).
   - Interpret the results and check if they meet physical expectations or validate with known solutions.

The FEA process is typically broken down into several distinct phases:

1. Pre-Processing (Model Setup)

In this phase, the problem is prepared for the analysis, and it involves several key steps:

- **Geometry Definition**: The first step is to define the geometry of the structure or system you are analyzing. This could be a solid, shell, or beam model depending on the application.
- **Material Properties**: Define the material properties for each element in the model. This can include properties like Young's Modulus, Poisson's ratio, density, thermal conductivity, etc.
- **Meshing**: The geometry is divided into small, simple elements (e.g., tetrahedra, hexahedra, or beam elements). Meshing involves selecting the type, size, and quality of these elements.
- **Boundary Conditions and Loads**: Define the external constraints and loads acting on the system. Boundary conditions could include fixed supports, symmetry constraints, or temperature variations, while loads might be forces, pressures, thermal gradients, etc.
- **Element Selection**: Choose the appropriate type of finite elements (e.g., 1D, 2D, 3D, solid, shell, or beam elements) based on the nature of the problem.

2. Solution Phase

Once the model is set up, the next step is to solve the system of equations that represent the physical behavior of the structure. This phase involves:

- **Assembly**: The global stiffness matrix is assembled from the individual element stiffness matrices. This process typically requires knowledge of the shape functions and interpolation methods.
- **Solving the System of Equations**: The governing equations (often in the form of linear or nonlinear equations) are solved. For linear problems, this can be done using methods like Gaussian elimination or iterative solvers. For nonlinear problems, more advanced techniques, such as Newton-Raphson, may be required.
- **Post-Processing for Nonlinear Problems**: For nonlinear problems, an iterative process is used, and the system is solved for each time step or increment.

3. **Post-Processing (Results Interpretation)**

In this phase, the results of the analysis are examined and interpreted:

- **Visualization**: Results are often visualized using contour plots, displacement plots, and deformation animations to better understand the behavior of the system.
- **Stress, Strain, and Deformation Analysis**: The main results are stresses, strains, deformations, and other derived quantities like factor of safety, heat distribution, etc.
- **Verification and Validation**: Results are checked for accuracy, often by comparing them to analytical solutions (for simple cases) or experimental data.
- **Optimization**: Based on the results, design changes might be proposed to improve performance, safety, or material usage.

4. **Post-Analysis Phase (Model Refinement and Reporting)**

This phase typically involves refining the model and finalizing the analysis:

- **Model Refinement**: Based on the results from the initial analysis, you might refine the mesh, update the material properties, or apply more accurate boundary conditions.
- **Sensitivity Analysis**: Perform a sensitivity analysis to understand how changes in input parameters (like material properties or load conditions) affect the results.
- **Documentation**: Finalize the analysis by preparing reports or presentation materials that summarize the model setup, assumptions, results, and any recommendations or design improvements.

Discretization process:

Discretization is a fundamental process in the Finite Element Method (FEM), which involves dividing a continuous domain (geometry or structure) into smaller, finite parts called elements. These elements are interconnected at specific points called nodes. By discretizing the domain, the governing partial differential equations (PDEs) of the problem can be approximated using a system of algebraic equations.

Steps in the Discretization Process:

1. **Geometric Domain Division**:
   - The first step is to break down the continuous geometry of the problem into a finite number of elements (e.g., triangles, quadrilaterals in 2D or tetrahedra, hexahedra in 3D).
   - The shape, size, and number of elements depend on the complexity of the geometry and the desired accuracy.
2. **Selection of Element Types**:
   - The type of elements used depends on the problem domain, e.g., line elements for 1D problems, triangular or quadrilateral elements for 2D, and tetrahedral or hexahedral elements for 3D problems.
   - Higher-order elements can be used for increased accuracy.
3. **Node Placement**:
   - Nodes are strategically placed at element corners, edges, or internally, depending on the type and order of the element.
   - Nodes serve as the points where the solution is explicitly computed.
4. **Interpolation Functions**:
   - Within each element, the solution is approximated using interpolation (or shape) functions that depend on nodal values.
   - These functions are typically linear or polynomial, depending on the order of the element.
5. **Governing Equation Approximation**:
   - The governing PDE is transformed into its weak or variational form, suitable for FEM application.
   - The domain is discretized, and the integral equations are approximated for each element.
6. **Assembly of Global System**:
   - The local element equations (stiffness matrix, force vector, etc.) are assembled into a global system of equations using connectivity information.
7. **Application of Boundary Conditions**:
   - Essential and natural boundary conditions are imposed on the discretized equations to ensure the solution satisfies physical constraints.
8. **Solving the System**:
   - The resulting system of algebraic equations is solved using numerical methods to obtain approximate solutions at the nodes.

Meshing-Element Type

In Finite Element Method (FEM), **meshing** is the process of dividing a geometric model into smaller, discrete elements to solve physical problems numerically. The type of **element** chosen during meshing depends on the geometry, problem type, and required accuracy.

Below are common **element types** used in FEM:

1. 1D Elements

- **Applications**: Beams, trusses, frames, and slender structures.
- **Types**:
  - **Bar/Truss Elements**: For axial forces only (e.g., tension, compression).
  - **Beam Elements**: For axial, bending, and shear forces.
- **Shape**: Straight or curved line segments.

2. 2D Elements

- **Applications**: Thin structures such as plates, shells, or planar problems (stress analysis in 2D).
- **Types**:
  - **Triangular (3-node or higher)**: Easier to mesh complex geometries; may be less accurate than quadrilaterals for the same mesh density.
  - **Quadrilateral (4-node or higher)**: Often preferred for better accuracy and efficiency in structured domains.
- **Shape**: Flat elements with triangular or quadrilateral geometry.

3. 3D Elements

- **Applications**: Solid mechanics, thermal problems, and 3D structures.
- **Types**:
  - **Tetrahedral (4-node or higher)**: Useful for complex geometries; easier to automate meshing.
  - **Hexahedral (8-node or higher)**: Better accuracy for structured domains but harder to mesh.
  - **Pyramidal and Wedge Elements**: Used for transitions between tetrahedral and hexahedral elements.
- **Shape**: Solid shapes such as tetrahedrons, hexahedrons, or wedges.

4. Shell Elements

- **Applications**: Thin-walled structures like aircraft fuselages, car bodies, or pipelines.
- **Types**:
  - **Linear Shells**: Simplified formulations, good for simple thin structures.
  - **Nonlinear Shells**: For large deformations and complex loading.

CHAPTER-10

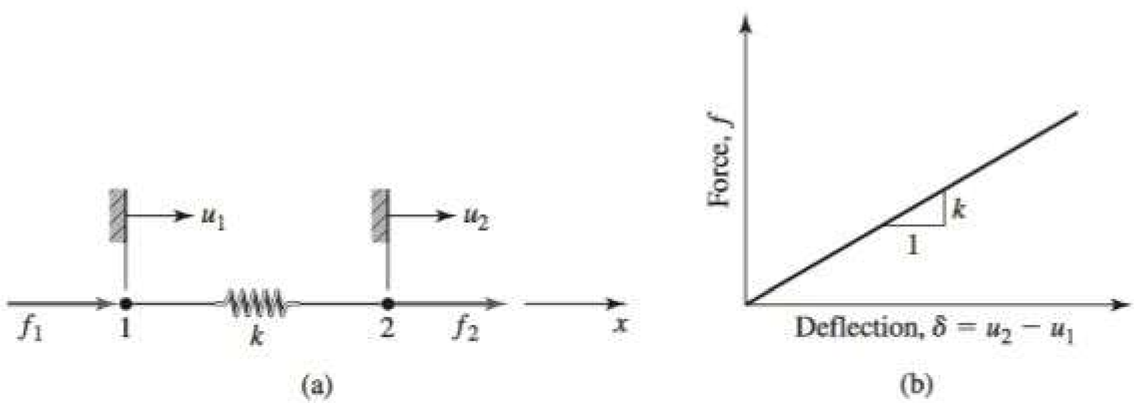
STIFNESS MATRIX

STIFFNESS MATRIX OF A BAR ELEMENT

The primary characteristics of a finite element are embodied in the element **stiffness matrix**. For a structural finite element, the stiffness matrix contains the geometric and material behavior information that indicates the resistance of the element to deformation when subjected to loading. Such deformation may include axial, bending, shear, and torsional effects. For finite elements used in nonstructural analyses, such as fluid flow and heat transfer, the term stiffness matrix is also used, since the matrix represents **the resistance of the element to change when subjected to external influences.**

Linear spring as a finite element

A linear elastic spring is a mechanical device capable of supporting axial loading only, and the elongation or contraction of the spring is directly proportional to the applied axial load. The constant of proportionality between deformation and load is referred to as the spring constant, spring rate, or **spring stiffness k**, and has units of force per unit length. As an elastic spring supports axial loading only, we select an element coordinate system (also known as a local coordinate system) as an x axis oriented along the length of the spring, as shown.



(a) Linear spring element with nodes, nodal displacements, and nodal forces.
(b) Load-deflection curve.

Assuming that both the nodal displacements are zero when the spring is undeformed, the net spring deformation is given by $\delta = u_2 - u_1$

and the resultant axial force in the spring is

$f = k\delta = k(u_2 - u_1)$

For equilibrium,

$f_1 + f_2 = 0$ or $f_1 = -f_2$,

Then, in terms of the applied nodal forces as

$\mathbf{f_1 = -k(u_2 - u_1)}$

$\mathbf{f_2 = k(u_2 - u_1)}$

which can be expressed in matrix form as

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \quad \text{or} \quad [k_e]\{u\} = \{f\}$$

where

$$\boxed{[k_e] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \qquad \textbf{Stiffness matrix for one spring element}}$$

is defined as the element stiffness matrix in the element coordinate system (or local system), {u} is the column matrix (vector) of nodal displacements, and { f } is the column matrix (vector) of element nodal forces.

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = [k_e] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \text{with} \quad [k_e] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

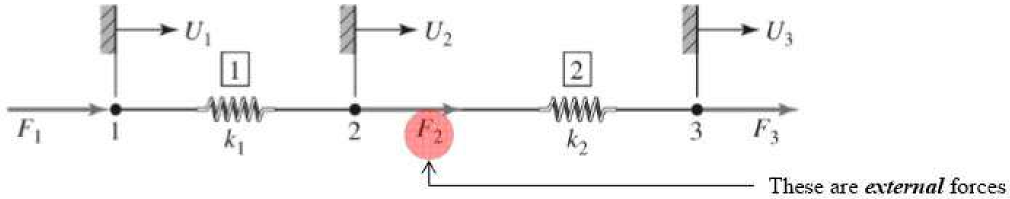
$$\text{known} \quad \{F\} = [K]\ \{X\} \quad \text{unknown}$$

The equation shows that the element stiffness matrix for the linear spring element is **a 2 × 2 matrix**. This corresponds to the fact that the element exhibits **two nodal displacements (or degrees of freedom) and that the two displacements are not independent (that is, the body is continuous and elastic).**

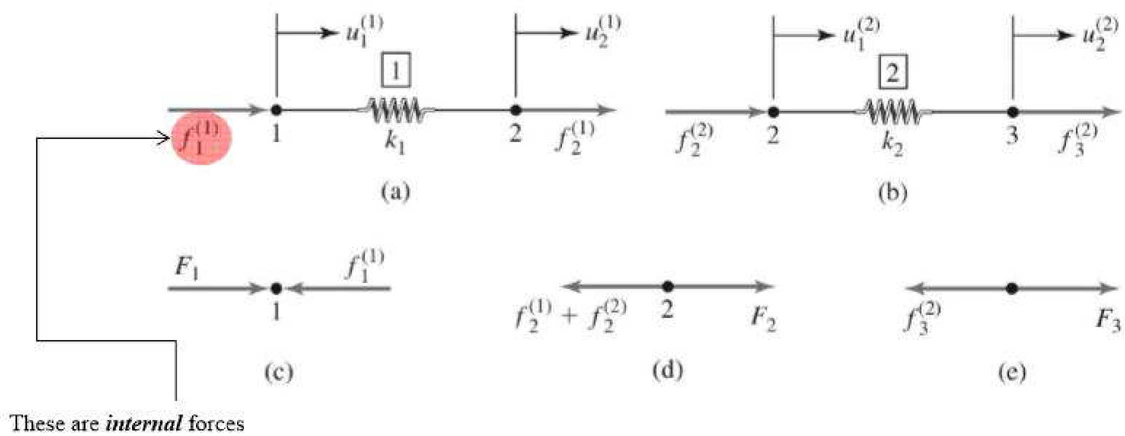
Furthermore, **the matrix is symmetric**. This is a consequence of the symmetry of the forces (equal and opposite to ensure equilibrium).

Also **the matrix is singular** and therefore not invertible. That is because the problem as defined is incomplete and does not have a solution: **boundary conditions are required**.

SYSTEM OF TWO SPRINGS [GLOBAL STIFFNESS MATRIX]



Free body diagrams:



Writing the equations for each spring in matrix form:

Superscript refers to element

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix}$$

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix}$$

To begin assembling the equilibrium equations describing the behavior of the system of two springs, the displacement **compatibility conditions**, which relate element displacements to system displacements, are written as:

$$u_1^{(1)} = U_1 \qquad u_2^{(1)} = U_2 \qquad u_1^{(2)} = U_2 \qquad u_2^{(2)} = U_3$$

And therefore:

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix}$$

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix}$$

Here, we use the notation $f^{(j)}{}_i$ to represent the force exerted on element j at node i.

Expand each equation in matrix form:

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix}$$

Summing member by member:

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix}$$

Next, we refer to the free-body diagrams of each of the three nodes:

$$f_1^{(1)} = F_1 \qquad f_2^{(1)} + f_2^{(2)} = F_2 \qquad f_3^{(2)} = F_3$$

Final form:

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \tag{1}$$

Where the stiffness matrix:

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

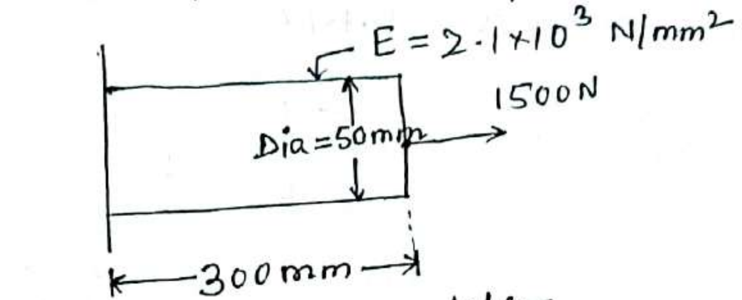
PROPERTIES OF STIFFNESS MATRIX

Note that the system stiffness matrix is:

(1) **symmetric**, as is the case with all linear systems referred to orthogonal coordinate systems;
(2) **singular**, since no constraints are applied to prevent rigid body motion of the system;

(3) the system matrix is simply **a superposition of the individual element stiffness matrices** with proper assignment of element nodal displacements and associated stiffness coefficients to system nodal displacements.

Q.1 A steel rod subjected to ~~reaction~~ tension is modelled by one 1D bar element, as shown in figure. Determine the nodal displacement, axial stress in each element and reaction forces. $E = 2.1\times10^3$ N/mm². poisson's ratio = 0.3.



Sol^n:- (i) Finite Element Model:-

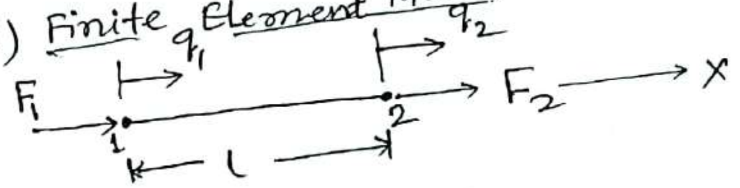


Fig:- Finite Element model of 1D bar element.

Given:- $E = 2.1\times10^3$ N/mm²

Area of element, $A = \frac{\pi}{4}D^2 = \frac{\pi}{4}(50)^2 = 1963.49$ mm²

Length of the element, $L = 300$ mm

Load at node 2, $F_2 = 1500$ N

(ii) Elemental Stiffness Matrix;

The stiffness matrix for the bar element (1D) is given by:-

$$[K_e] = \frac{A_e E_e}{l_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

for Element 1,

$$[K_1] = \frac{A_1 E_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$[K_1] = \frac{1963.49 \times 2.1 \times 10^5}{300} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 13.74 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(iii) Global stiffness Matrix

$$[K] = [K_1] = 13.74 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Global nodal displacement vector

$$\{Q\} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

Global load vector

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1500 \end{Bmatrix}$$

(iv) Equilibrium condition:-

$$\{F\} = [K]\{Q\}$$

$$\begin{Bmatrix} 0 \\ 1500 \end{Bmatrix} = 13.74 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

(v) Applying boundary condition:-

$q_1 = 0$ (Fixed end)

$$\therefore \begin{Bmatrix} 0 \\ 1500 \end{Bmatrix} = 13.74 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ q_2 \end{Bmatrix}$$

or $1500 = 13.74 \times 10^5 \times q_2$

or $q_2 = 0.001091$ mm.

$\therefore$ The nodal displacement vector is $\{Q\} = \begin{Bmatrix} 0 \\ 0.001091 \end{Bmatrix}$ mm.

(vi) Stress in each element :-

$$\sigma = \frac{E}{l}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}q_1\\q_2\end{Bmatrix}$$

$$= \frac{2.1\times10^5}{300}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}0\\0.001091\end{Bmatrix}$$

$$= 700\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}0\\0.001091\end{Bmatrix}$$

$$= \begin{bmatrix}-700 & 700\end{bmatrix}\begin{Bmatrix}0\\0.001091\end{Bmatrix}$$

$$= \begin{bmatrix}-700\times0 & 700\times0.001091\end{bmatrix}$$

$$\sigma = 0.7637 \text{ N/mm}^2 \quad \underline{\text{Ans}}$$

(vii) Reaction at support:

$$\{R\} = [K]\{Q\} - \{F\}$$

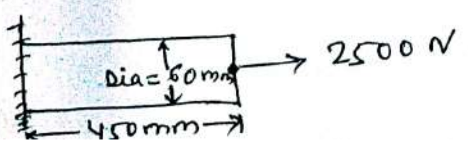
$$\begin{Bmatrix}R_1\\R_2\end{Bmatrix} = 13.74\times10^5\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}0\\0.001091\end{Bmatrix} - \begin{Bmatrix}0\\1500\end{Bmatrix}$$

$$\begin{Bmatrix}R_1\\R_2\end{Bmatrix} = \begin{Bmatrix}-1499.03\\1499.03\end{Bmatrix} - \begin{Bmatrix}0\\1500\end{Bmatrix}$$

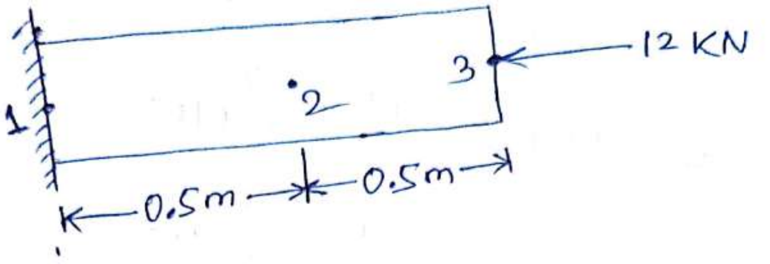
$$\begin{Bmatrix}R_1\\R_2\end{Bmatrix} = \begin{Bmatrix}-1499.03\\0\end{Bmatrix}\text{N} \quad \{\text{Since } 1499\text{ N} \simeq 1500\text{ N}\}$$

$$\begin{Bmatrix}R_1\\R_2\end{Bmatrix} = \begin{Bmatrix}-1500\\0\end{Bmatrix}\text{N} \quad \underline{\text{Ans}}$$

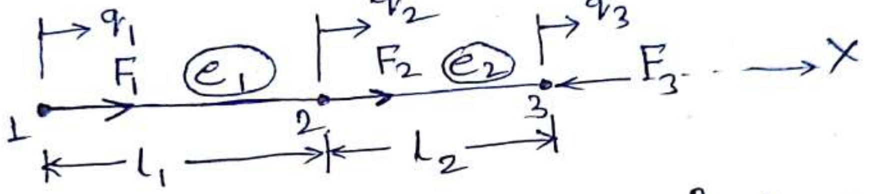
Q2 A steel rod subjected to tension is modelled by one bar element as shown in figure. Determine the nodal displacement and the axial stress in each element and reaction forces. $E = 2.1\times10^5$ N/mm², Poisson's ratio = 0.3



Q3 A steel rod subjected to compression is modelled by two bar elements as shown in figure. Determine the nodal displacements and axial stress in each element.
Take $E = 207\ GPa$, $A = 500\ mm^2$.



Sol^n:- (i) Finite Element (FE) Model:



$$E = E_1 = E_2 = 207\ GPa = 207 \times 10^3\ N/mm^2$$

$$A = A_1 = A_2 = 500\ mm^2$$

$$L = l_1 = l_2 = 0.5\,m = 500\,mm$$

$F_3 = -12\,KN$ (opposite to x-direction)

(ii) Elemental stiffness Matrix:

$$[K_e] = \frac{A_e E_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For element 1,

$$[K_1] = \frac{A_1 E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{500 \times 207 \times 10^3}{500}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^3\begin{bmatrix} 207 & -207 \\ -207 & 207 \end{bmatrix}$$

For element 2;

$$[K_2] = \frac{A_2 E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_2] = \frac{500 \times 207 \times 10^3}{500} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_2] = 10^3 \begin{bmatrix} 207 & -207 \\ -207 & 207 \end{bmatrix}$$

(iii) Global Stiffness matrix:

$$[K] = [K_1] + [K_2]$$

$$[K] = 10^3 \begin{bmatrix} 207 & -207 & 0 \\ -207 & 414 & -207 \\ 0 & -207 & 207 \end{bmatrix}$$

Global nodal displacement vector;

$$\{Q\} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

Global load vector;

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -12000 \end{Bmatrix}$$

(iv) equilibrium condition:-

$$\{F\} = [K]\{Q\}$$

$$\begin{Bmatrix} 0 \\ 0 \\ -12000 \end{Bmatrix} = 10^3 \begin{bmatrix} 207 & -207 & 0 \\ -207 & 414 & -207 \\ 0 & -207 & 207 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

(v) Applying boundary conditions;

$q_1 = 0$ (fixed end).

$$\begin{Bmatrix} 0 \\ -12000 \end{Bmatrix} = 10^3 \begin{bmatrix} 414 & -207 \\ -207 & 207 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}$$

on simplification, we get.

$$414\times10^3 q_2 - 207\times10^3 q_3 = 0$$
$$-207\times10^3 q_2 + 207\times10^3 q_3 = -12000$$

$\therefore q_2 = -0.05797101014$ mm.

$$\boxed{q_2 = -0.05797 \text{ mm}}$$

$q_3 = -0.11594$ mm.

$\therefore$ Nodal displacement vector is

$$\{q\} = \begin{Bmatrix} 0 \\ -0.05797 \\ -0.11594 \end{Bmatrix} \text{mm.}$$

(vi) Stress in each element:,

$$\sigma = \frac{E_e}{l_e}[-1 \;\; 1]\begin{Bmatrix} q_i \\ q_{i+1} \end{Bmatrix}.$$

Where, e = Element number
i = Node number.

<u>for Element 1</u>

$$\sigma_1 = \frac{E_1}{l_1}[-1 \;\; 1]\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$\sigma_1 = \frac{207\times10^3}{500}[-1 \;\; 1]\begin{Bmatrix} 0 \\ -0.05797 \end{Bmatrix}$$

$$= -23.99958 \text{ N/mm}^2$$

<u>For Element 2:</u>

$$\sigma_2 = \frac{207\times10^3}{500}[-1 \;\; 1]\begin{Bmatrix} -0.05797 \\ -0.11594 \end{Bmatrix}$$

$$= -23.99958 \text{ N/mm}^2$$

$$\boxed{\sigma_2 = -24 \text{ N/mm}^2}$$

(vii) Reaction at support :-

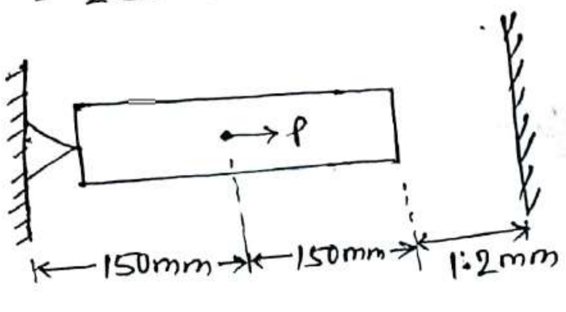
$$\{R\} = [K]\{Q\} - \{F\}$$

$$= 10^3 \begin{bmatrix} 207 & -207 & 0 \\ -207 & 414 & -207 \\ 0 & -207 & 207 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.05797 \\ -0.11594 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ -12000 \end{Bmatrix}$$

After simplification,

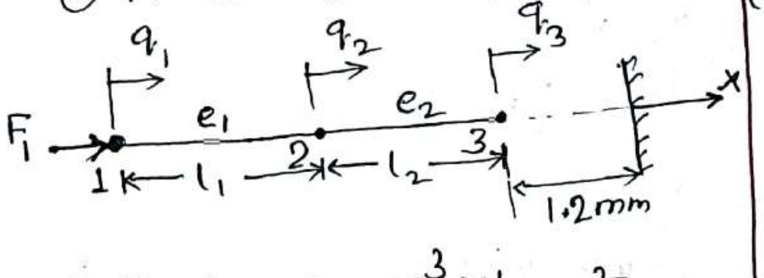
$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} 12000 \\ 0 \\ 0 \end{Bmatrix} N \quad \underline{\underline{\text{Ans}}}$$

Q4 A load of $P = 60$ KN is applied as shown in fig. Determine the following :-
(a) Nodal displacement, (b) stress in each member, (c) Reaction forces.
Given :- $E = 20 \times 10^3$ N/mm², poisson's ratio, $\mu = 0.3$
$A = 250$ mm².



Solution :-

(1) Finite Element Model :-



$E = E_1 = E_2 = 20 \times 10^3$ N/mm²

$A = A_1 = A_2 = 250$ mm²

$l_1 = l_2 = 150$ mm

$F_2 = 60 \times 10^3$ N

(2) Elemental Stiffness Matrix

The stiffness matrix for the bar element is given by

$$[K_e] = \frac{A_e E_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For element 1,

$$[K_1] = \frac{250 \times 20 \times 10^3}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 0.33 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For element 2

$$[K_2] = \frac{250 \times 20 \times 10^3}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 0.33 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(iii) Global stiffness matrix

$$[K] = [K_1] + [K_2]$$

$$= 0.33 \times 10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Global Nodal displacement vector

$$\{Q\} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

Global load vector

$$\{F\} = \begin{Bmatrix} (F_1 - K_{13} q_3) \\ (F_2 - K_{23} q_3) \\ (F_3 - K_{33} q_3) \end{Bmatrix}$$

Where,

q_3 = Max. displacement = 1.2 mm

④ Equilibrium condition

$$[K]\{Q\} = \{F\}$$

Substituting the values of [K], {Q}, {F}, we get,

$$10^5 \begin{bmatrix} 0.33 & -0.33 & 0 \\ -0.33 & 0.66 & -0.33 \\ 0 & -0.33 & 0.33 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 - K_{13} q_3 \\ F_2 - K_{23} q_3 \\ F_3 - K_{33} q_3 \end{Bmatrix}$$

⑤ Applying boundary conditions

From figure, $q_1 = 0$, $q_3 = 1.2$ mm.

$$10^5 \begin{bmatrix} 0.33 & -0.33 & 0 \\ -0.33 & 0.66 & -0.33 \\ 0 & -0.33 & 0.33 \end{bmatrix} \begin{Bmatrix} 0 \\ q_2 \\ 1.2 \end{Bmatrix} = \begin{Bmatrix} 0 - 0 \times 1.2 \times 10^5 \\ 60 \times 10^3 + (0.33) \times 1.2 \times 10^5 \\ 0 - 0.33 \times 1.2 \times 10^5 \end{Bmatrix}$$

where,

$K_{13} = 0 \times 10^5$

$K_{23} = -0.33 \times 10^5$

$K_{33} = 0.33 \times 10^5$

On simplification, we get

$q_2 = 1.50909$ mm

The nodal displacement vector

$$\{Q\} = \begin{Bmatrix} 0 \\ 1.50909 \\ 1.2 \end{Bmatrix} \text{ mm} \quad \underline{\underline{\text{Ans}}}$$

(6) Stress in each element

For element 1,

$$\sigma_1 = \frac{E_1}{L_1}[-1\ 1]\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \frac{20\times10^3}{150}[-1\ 1]\begin{Bmatrix} 0 \\ 1.50909 \end{Bmatrix} = 201.20\ \text{N/mm}^2$$

For element 2,

$$\sigma_2 = \frac{E_2}{L_2}[-1\ 1]\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \frac{20\times10^3}{150}[-1\ 1]\begin{Bmatrix} 1.50909 \\ 1.2 \end{Bmatrix} = -41.204\ \text{N/mm}^2$$

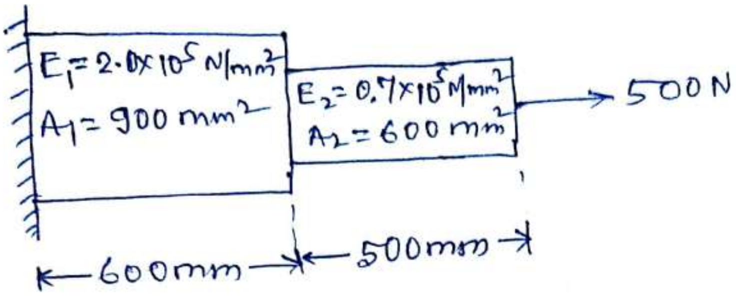
(7) Support reaction:

$$\{R\} = [K]\{Q\} - \{F\}$$

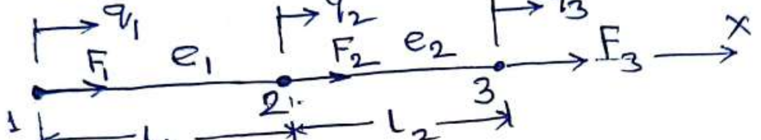
$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 10^5\begin{bmatrix} 0.33 & -0.33 & 0 \\ -0.33 & 0.66 & -0.33 \\ 0 & -0.33 & 0.33 \end{bmatrix}\begin{Bmatrix} 0 \\ \cancel{60\times10^3}\ 1.50909 \\ 1.2 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 60\times10^3 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} -49790 \\ 0 \\ -10190 \end{Bmatrix} \text{ N} \quad \underline{\underline{\text{Ans}}}$$

Q5. Consider the stepped bar as shown in figure below. Determine the nodal displacement, stress in each element and reaction forces.



Soln:- ① Finite Element model:-



Data:- $E_1 = 2.0\times10^5$ N/mm², $A_1 = 900$ mm², $l_1 = 600$ mm

$E_2 = 0.7\times10^5$ N/mm², $A_2 = 600$ mm², $l_2 = 500$ mm

② Elemental stiffness matrix:-

The stiffness matrix for the bar element is given by

$$[K_e] = \frac{A_e E_e}{l_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

for element 1, $[K_1] = \frac{900\times 2.0\times10^5}{600}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 10^5\begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}$

for element 2, $[K_2] = \frac{600\times 0.7\times10^5}{500}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 10^5\begin{bmatrix}0.84 & -0.84\\ -0.84 & 0.84\end{bmatrix}$

③ Global stiffness matrix ~~is~~

The global stiffness matrix for stepped bar is given by

$$[K] = [K_1] + [K_2] = 10^5\begin{bmatrix}3 & -3 & 0\\ -3 & 3.84 & -0.84\\ 0 & -0.84 & 0.84\end{bmatrix}$$

Global load vector

$$\{q\} = \begin{Bmatrix}q_1\\ q_2\\ q_3\end{Bmatrix}$$

Global load vector

$$F = \begin{Bmatrix}F_1\\ F_2\\ F_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ 500\end{Bmatrix}$$

(4) Equilibrium Condition

$[K]\{Q\} = \{F\}$

$$10^5 \begin{bmatrix} 3 & -3 & 0 \\ -3 & 3.84 & -0.84 \\ 0 & -0.84 & 0.84 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 500 \end{Bmatrix}$$

(5) Applying boundary condition:-

$q_1 = 0$ (fixed end)

$$10^5 \begin{bmatrix} 3.84 & -0.84 \\ -0.84 & 0.84 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 500 \end{Bmatrix}$$

$3.84 \times 10^5 q_2 - 0.84 \times 10^5 q_3 = 0$ —(i)

$-0.84 \times 10^5 q_2 + 0.84 \times 10^5 q_3 = 500$ —(ii)

$$q_2 = \frac{500}{3 \times 10^5} = 0.001667 \text{ mm}$$

$$q_3 = 0.00762057 \text{ mm}$$

∴ The nodal displacement vector is:- $\{Q\} = \begin{Bmatrix} 0 \\ 0.001667 \\ 0.007620 \end{Bmatrix}$ Ans

(6) Stress in each element:-

$$\sigma = \frac{E_e}{l_e} [-1 \ 1] \begin{Bmatrix} q_i \\ q_{i+1} \end{Bmatrix}$$

Where ~~i = e~~ e = element number

i = node number.

For element 1

$$\sigma_1 = \frac{2 \times 10^5}{600} [-1 \ 1] \begin{Bmatrix} 0 \\ 0.001667 \end{Bmatrix}$$

$\sigma_1 = 0.555$ N/mm² Ans

For element 2

$$\sigma_2 = \frac{0.7 \times 10^5}{500} [-1 \ 1] \begin{Bmatrix} 0.001667 \\ 0.007620 \end{Bmatrix}$$

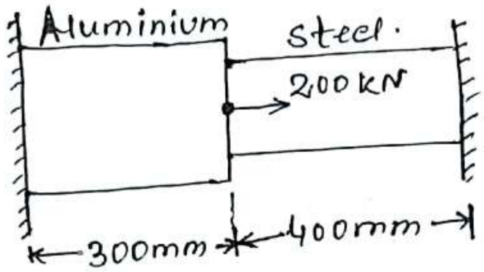
$\sigma_2 = 0.83342$ N/mm² Ans

(7) Reaction at support:-

$$\{R\} = [K]\{Q\} - \{F\} = 10^5 \begin{bmatrix} 3 & -3 & 0 \\ -3 & 3.84 & -0.84 \\ 0 & -0.84 & 0.84 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.001667 \\ 0.007620 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 500 \end{Bmatrix}$$

After simplification, we get, $\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} 500 \\ 0 \\ 0 \end{Bmatrix}$ N Ans

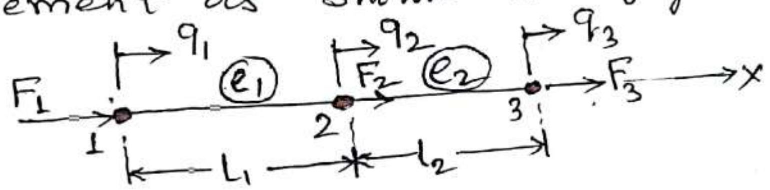
Q.6 Find nodal displacement, stress in each element and reaction forces for the following problem. Given $E_1 = 70\times10^3\ N/mm^2$, $E_2 = 200\times10^3\ N/mm^2$, $A_1 = 2400\ mm^2$, $A_2 = 600\ mm^2$, $\gamma = 0.3$.



Solution:-

① FE Model:-

Modelling the given stepped bar using two bar element as shown in figure:-



Data: $E_1 = 70\times10^3\ N/mm^2$, $E_2 = 200\times10^3\ N/mm^2$

$A_1 = 2400\ mm^2$, $A_2 = 600\ mm^2$.

$l_1 = 300\ mm$, $l_2 = 400\ mm$.

② Elemental stiffness matrix:

The stiffness matrix for the stepped bar is given by.

$$[K_e] = \frac{A_e E_e}{l_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

For element 1

$$[K_1] = \frac{A_1E_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{2400\times70\times10^3}{300}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 10^5\begin{bmatrix}5.6 & -5.6\\ -5.6 & 5.6\end{bmatrix}$$

For element 2,

$$[K_2] = \frac{A_2E_2}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{600\times200\times10^3}{400}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 10^5\begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}$$

③ Global stiffness matrix:

The global stiffness matrix for stepped bar is given by

$$[K] = [K_1]+[K_2] = 10^5\begin{bmatrix}5.6 & -5.6 & 0\\ -5.6 & 8.6 & -3\\ 0 & -3 & 3\end{bmatrix}$$

Global nodal displacement vector :- $\{Q\} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$

Global load vector: $\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 200\times10^3 \\ 0 \end{Bmatrix}$

④ Equilibrium condition:.

$[K]\{Q\} = \{F\}$

Substituting the values of $[K]$, $\{Q\}$ and $\{F\}$ we get,

$$10^5\begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 200\times10^3 \\ 0 \end{Bmatrix}$$

⑤ Applying boundary conditions

From figure; $q_1 = 0$ {due to fixed end}
$q_3 = 0$

Note:- Using elimination method of handling boundary conditions, eliminating corresponding row and column with respect to the known boundary condition, i.e. $q_1 = 0$ & $q_3 = 0$, we get.

$$8.6\times10^5\, q_2 = 200\times10^3$$

$$q_2 = 0.232558 \text{ mm}$$

The nodal displacement vector is

$$\{Q\} = \begin{Bmatrix} 0 \\ 0.232558 \\ 0 \end{Bmatrix} \text{ mm} \quad \underline{\underline{\text{Ans}}}$$

⑥ Stress in each element

The stress in a 1D bar element is given by—

$$\sigma_e = \frac{E_e}{l_e}[-1 \;\; 1]\begin{Bmatrix} q_i \\ q_{i+1} \end{Bmatrix}$$

Where $i = e$ i = node number
e = no. of element

For element 1,

$$\sigma_1 = \frac{E_1}{l_1}[-1 \;\; 1]\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \frac{70\times10^3}{300}[-1 \;\; 1]\begin{Bmatrix} 0 \\ 0.232558 \end{Bmatrix} = 54.26 \text{ N/mm}^2$$

$\underline{\underline{\text{Ans}}}$

For element 2,

$$\sigma_2 = \frac{E_2}{L_2}[-1\ 1]\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \frac{200\times10^5}{400}[-1\ 1]\begin{Bmatrix} 0.23255 \\ 0 \end{Bmatrix} = -116.275\ N/mm^2$$ Ans

⑦ Reaction force at support:

The reaction force at support is given by-

$$\{R\} = [K]\{Q\} - \{F\}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 10^5\begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.23255 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 200\times10^3 \\ 0 \end{Bmatrix}$$

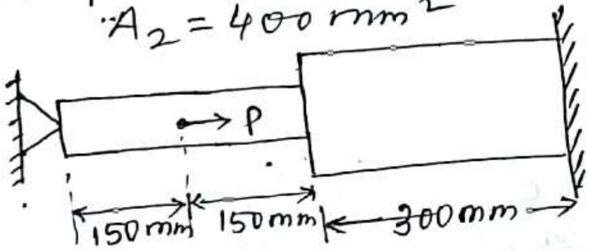
After Simplification, we get

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} -130228 \\ 0 \\ -69765 \end{Bmatrix} N$$ Ans

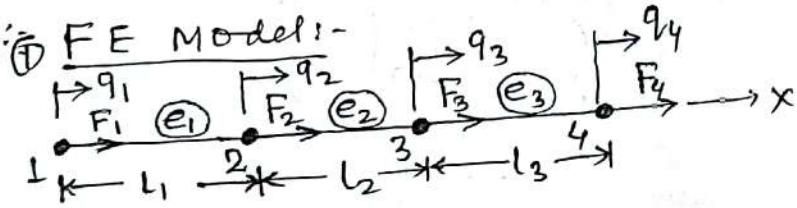
Q.7. Consider the bar loaded as shown in figure. Determine the nodal displacement, stress in each element and reaction forces.

Given: $P = 300\ kN$, $E = 200\times10^3\ N/mm^2$, $\gamma = 0.3$, $A_1 = 250\ mm^2$

$A_2 = 400\ mm^2$



Sol^n: ① FE Model:-



Data: $P = 300\ kN$

$E = E_1 = E_2 = E_3 = 200\times10^3\ N/mm^2$

$A = A_1 = A_2 = 250\ mm^2$ $\quad A_3 = 400\ mm^2$

$L_1 = L_2 = 150\ mm$, $L_3 = 300\ mm$.

② Elemental stiffness matrix:

$$[K_e] = \frac{A_e E_e}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For element 1,

$$[K_1] = \frac{250 \times 200 \times 10^3}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{bmatrix} 3.33 & -3.33 \\ -3.33 & 3.33 \end{bmatrix}$$

Similarly for element 2,

$$[K_2] = \frac{250 \times 200 \times 10^3}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{bmatrix} 3.33 & -3.33 \\ -3.33 & 3.33 \end{bmatrix}$$

Similarly for element 3,

$$[K_3] = \frac{400 \times 200 \times 10^3}{300} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{bmatrix} 2.67 & -2.67 \\ -2.67 & 2.67 \end{bmatrix}$$

③ Global stiffness Matrix:-

$$[K] = [K_1] + [K_2] + [K_3] = 10^5 \begin{bmatrix} 3.33 & -3.33 & 0 & 0 \\ -3.33 & 6.66 & -3.33 & 0 \\ 0 & -3.33 & 6 & -2.67 \\ 0 & 0 & -2.67 & 2.67 \end{bmatrix}$$

Global Nodal displacement vector:-

$$\{Q\} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

Global load vector:- $\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 300 \times 10^3 \\ 0 \\ 0 \end{Bmatrix}$

④ Equilibrium condition:-

$$\{F\} = [K]\{Q\}$$

$$\begin{Bmatrix} 0 \\ 300 \times 10^3 \\ 0 \\ 0 \end{Bmatrix} = 10^5 \begin{bmatrix} 3.33 & -3.33 & 0 & 0 \\ -3.33 & 6.66 & -3.33 & 0 \\ 0 & -3.33 & 6 & -2.67 \\ 0 & 0 & -2.67 & 2.67 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

⑤ Applying boundary conditions:-

From figure -

$q_1 = 0$
$q_4 = 0$ } due to fixed end}

$$\begin{bmatrix} 300 \times 10^3 \\ 0 \end{bmatrix} = 10^5 \begin{bmatrix} 6.66 & -3.33 \\ -3.33 & 6 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}$$ [Using elimination Method]

on simplification, we get

$q_2 = 0.62346$ mm and $q_3 = 0.34602$ mm

The displacement vector is

$$\{Q\} = \begin{Bmatrix} 0 \\ 0.62346 \\ 0.34602 \\ 0 \end{Bmatrix} \text{mm} \quad \underline{\text{Ans}}$$

⑥ The stress in each element:

For element 1,

$$\sigma_1 = \frac{E_1}{l_1}[-1 \ 1]\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \frac{200\times10^3}{150}[-1 \ 1]\begin{Bmatrix} 0 \\ 0.62346 \end{Bmatrix} = 831.2 \text{ N/mm}^2$$

For element 2,

$$\sigma_2 = \frac{E_2}{l_2}[-1 \ 1]\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \frac{200\times10^3}{150}[-1 \ 1]\begin{Bmatrix} 0.62346 \\ 0.34602 \end{Bmatrix} = -369.91 \text{ N/mm}^2$$

For element 3,

$$\sigma_3 = \frac{E_3}{l_3}[-1 \ 1]\begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \frac{200\times10^5}{300}[-1 \ 1]\begin{Bmatrix} 0.34602 \\ 0 \end{Bmatrix} = -230.68 \text{ N/mm}^2$$

⑦ Support reaction:

$$\{R\} = [K]\{Q\} - \{F\}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = 10^5 \begin{bmatrix} 3.33 & -3.33 & 0 & 0 \\ -3.33 & 6.66 & -3.33 & 0 \\ 0 & -3.33 & 6 & -2.67 \\ 0 & 0 & -2.67 & 2.67 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.62346 \\ 0.34602 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 300\times10^3 \\ 0 \\ 0 \end{Bmatrix}$$

On simplification, we get

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{Bmatrix} -207612.18 \\ 0 \\ -0.1800 \\ -92387.34 \end{Bmatrix} \approx \begin{Bmatrix} -207612.18 \\ 0 \\ 0 \\ -92387.34 \end{Bmatrix} \quad \underline{\text{Ans}}$$

Chapter No.- 13

Problems on 2-D Element of FEM

(a) 3-node linear (b) 6-Node quadratic (c) 9-Node cubic

Figure : Triangular Elements

Basic formulas and matrix to solve 2D elements:

① Area of the CST (constant strain triangle) 2D triangular element

$$A_e = \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

② Stiffness matrix of 2D CST triangular element

$$[K_e] = [B]^T[D][B] \times t \times A_e$$

Where, $[B]$ = strain displacement matrix

$[D]$ = Material constant matrix, which depends upon whether the given problem is plane stress or plane strain condition.

t = Thickness of the triangular element.

A_e = cross section area of the element

Strain displacement matrix is given by

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

Where, $\beta_1 = y_2 - y_3$, $\beta_2 = y_3 - y_1$, $\beta_1 = y_1 - y_2$

$\gamma_1 = -(x_2 - x_3)$, $\gamma_2 = -(x_3 - x_1)$, $\gamma_1 = -(x_1 - x_2)$

x and y are the co-ordinates.

strain matrix

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

③ Elemental ~~stiffness~~ stress matrix

The stress in 2D element is given by

$$\sigma = D\,\varepsilon$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = D \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

for plain stress, $D = \frac{E}{1-\gamma^2} \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & \frac{1-\gamma}{2} \end{bmatrix}$

for plain strain, $D = \frac{E}{(1-\gamma)(1-2\gamma)} \begin{bmatrix} 1-\gamma & \gamma & 0 \\ \gamma & 1-\gamma & 0 \\ 0 & 0 & \frac{1-2\gamma}{2} \end{bmatrix}$

where, γ = poission's ratio

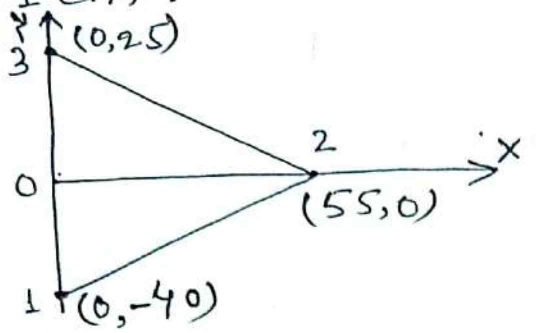
④ Reaction force:

$$[K]\{U\} = \{F\}$$

where $[K] = [B^T][D][B] \times t \times A_e$

$$\{U\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \qquad F = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$

Q.1 Evaluate the stiffness matrix for the CST 2D element shown below: ~~Coordi~~ coordinates in mm. Assume plain stress conditions.
$E = 200\,GPa$, $\gamma = 0.03$ (Poisson's ratio), Thickness $= 1\,cm$.



Nodal displacement are given as
$u_1 = 1mm$, $u_2 = 0.5mm$,
$u_3 = 2mm$, $v_1 = 1mm$,
$v_2 = 0$, $v_3 = 1mm$
Also find the stress in the element.

<u>Solution:-</u>

$(x_1, y_1) = (0, -40)$; $(x_2, y_2) = (55, 0)$; $(x_3, y_3) = (0, 25)$
$(x_1, y_1) = (0, -4)$; $(x_2, y_2) = (5.5, 0)$; $(x_3, y_3) = (0, 2.5)$
(for simplification converted the coordinates of CST 2D elements from mm to cm)

$\beta_1 = y_2 - y_3 = 0 - 2.5 = -2.5$
$\beta_2 = (y_3 - y_1) = 2.5 - (-4) = 6.5$
$\beta_3 = (y_1 - y_2) = -4 - 0 = -4$
$\gamma_1 = -(x_2 - x_3) = -(5.5 - 0) = -5.5$
$\gamma_2 = -(x_3 - x_1) = -(0 - 0) = 0$
$\gamma = -(x_1 - x_2) = -(0 - 5.5) = 5.5$

Let A be the area of triangle.

$$A = \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -4 \\ 1 & 5.5 & 0 \\ 1 & 0 & 2.5 \end{bmatrix} = 17.875\ cm^2$$

stiffness matrix is given by

$$[K] = [B^T][D][B] \times t \times A \text{ ------ (i)}$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$= \frac{1}{2\times 17.875}\begin{bmatrix} -2.5 & 0 & 6.5 & 0 & -4 & 0 \\ 0 & -5.5 & 0 & 0 & 0 & 5.5 \\ -5.5 & -2.5 & 0 & 6.5 & 5.5 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -0.07 & 0 & 0.182 & 0 & -0.112 & 0 \\ 0 & -0.154 & 0 & 0 & 0 & 0.154 \\ 0.154 & -0.07 & 0 & 0.182 & 0.154 & -0.112 \end{bmatrix}$$

$$[B^T] = \begin{bmatrix} -0.07 & 0 & -0.154 \\ 0 & -0.154 & -0.07 \\ 0.182 & 0 & 0 \\ 0 & 0 & 0.182 \\ -0.112 & 0 & 0.154 \\ 0 & 0.154 & -0.112 \end{bmatrix}$$

$$[D] = \frac{E}{1-\gamma^2}\begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & \frac{1-\gamma}{2} \end{bmatrix}$$ (for plain stress conditions)

$$= \frac{2\times 10^7}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

$$= 21.98\times 10^6\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

Now substitute the values of $[B]$, $[D]$, $[B^T]$, t, A in (i)

$$[K] = \begin{bmatrix} -0.07 & 0 & -0.154 \\ 0 & -0.154 & -0.07 \\ 0.182 & 0 & 0 \\ 0 & 0 & 0.182 \\ -0.112 & 0 & 0.154 \\ 0 & 0.154 & -0.112 \end{bmatrix} \times 21.98\times 10^6 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \times \begin{bmatrix} -0.07 & 0 & 0.182 & 0 & -0.112 & 0 \\ 0 & -0.154 & 0 & 0 & 0 & 0.154 \\ -0.154 & 0 & 0 & 0.182 & 0.154 & -0.112 \end{bmatrix} \times 1 \times 17.875$$

$$[K] = 3929\times10^6\begin{bmatrix} 0.132 & 0.07 & -0.0127 & -0.0098 & -0.0035 & -0.0028 \\ 0.007 & 0.0255 & -0.0084 & -0.0045 & 0.0014 & -0.021 \\ -0.0127 & -0.0084 & 0.033 & 0 & -0.0203 & 0.0084 \\ -0.0098 & -0.0045 & 0 & 0.0116 & 0.0098 & -0.0071 \\ 0.0035 & 0.0014 & -0.0203 & 0.0078 & 0.0208 & 0.0112 \\ -0.0028 & -0.021 & 0.0084 & -0.0071 & -0.0112 & -0.0282 \end{bmatrix}$$

Reaction force calculation:

$$[K]\{U\} = \{F\}$$

$$10^6\begin{bmatrix} 5.19 & 2.75 & -5 & -3.84 & -0.181 & 1.1 \\ 2.75 & 10 & -3.3 & -1.77 & 0.55 & -8.25 \\ -5 & -3.3 & 13 & 0 & -8 & 3.3 \\ -3.85 & -1.77 & 0 & 4.56 & 3.85 & -2.79 \\ -0.181 & 0.55 & -8 & 3.85 & 8.17 & -4.4 \\ 1.1 & -8.25 & 3.3 & -2.79 & -4.4 & 11 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$

Apply boundary conditions:-

$u_1 = 0.1\,cm,\ v_2 = 0.1\,cm,\ u_2 = 0.05\,cm,\ v_2 = 0,\ u_3 = 0.2\,cm$

$v_3 = 0.1\,cm$

After simplification, we get

$F_{1x} = 617.8\,KN,\ F_{1y} = 395\,KN,\ F_{2x} = -1450\,KN,\ F_{2y} = -71\,KN$

$F_{3x} = 830.9\,KN,\ F_{3y} = -330\,KN$

Elemental stress matrix

$$\{\sigma\} = [D]\{\varepsilon\}.$$
$$= [D][B]\{U\}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D]\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} -0.071 & 0 & 0.182 & 0 & -0.112 & 0 \\ 0 & -1.54 & 0 & 0 & 0 & 0.154 \\ 0.154 & -0.07 & 0 & 0.182 & 0.154 & -0.112 \end{bmatrix} \times \begin{Bmatrix} 0.1 \\ 0.1 \\ 0.05 \\ 0 \\ 0.2 \\ 0.1 \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = 10^{-3}\begin{Bmatrix} -20.4 \\ 0 \\ 2.8 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 21.98\times10^6\times10^{-3}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{bmatrix} -20.4 \\ 0 \\ -2.8 \end{bmatrix}$$

After simplification, we get

$\sigma_x = -4483.5\,N/mm^2,\ \sigma_y = -1345.05\,N/mm^2,\ \tau_{xy} = -215.38\,N/mm^2$